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Primitive permutation groups with a regular subgroup

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Abstract

This paper starts the classification of the primitive permutation groups (G, Ω) such that G contains a regular subgroup X. We determine all the triples (G, Ω, X) with $\mathrm{soc}(G)$ an alternating, or a sporadic or an exceptional group of Lie type. Further, we construct all the examples (G, Ω, X) with G a classical group which are known to us. Our particular interest is in the 8-dimensional orthogonal groups of Witt index 4. We determine all the triples (G, Ω, X) with $\mathrm{soc}(G) \cong \mathrm{P}\Omega_8^+(q)$. In order to obtain all these triples, we also study the almost simple groups G with $G \cong \mathrm{P}\Omega_{2n+1}(q)$. The case $G \cong \mathrm{U}_n(q)$ is started in this paper and finished in [B. Baumeister, Primitive permutation groups of unitary type with a regular subgroup, Bull. Belg. Math. Soc. 112 (5) (2006) 657–673]. A group X is called a *Burnside-group* (or short a *B-group*) if each primitive permutation group which contains a regular subgroup isomorphic to X is necessarily 2-transitive. In the end of the paper we discuss B-groups.

Keywords: Permutation group; Primitive permutation group; Regular subgroup; Almost simple group; Simple group; Classical group; Sporadic group; Orthogonal group; Factorization; Permutation representation; Burnside-group; B-group

1. Introduction

Suppose that (G, Ω) is a permutation group G acting on a set Ω. A subgroup U of G is called *regular* if for every pair (α, β) of elements in Ω there is precisely one element u in U which maps α onto β. For many applications it is necessary to know if G has a regular subgroup or not, and, if possible, to know all the subgroups which are regular.

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A permutation group is made up of primitive permutation groups. Therefore, it is reasonable to solve the problem first for the primitive permutation groups. The ambition of this paper is to start the classification of the primitive permutation groups (G,Ω) such that G contains a regular subgroup X.

If this is completed, it will imply, for instance, the complete list of the primitive graphs which are Cayley graphs (for an introduction to Cayley graphs see for instance [15]). Another application is that one can construct a semisimple Hopf algebra from these data, see for instance [36] or [13]. Notice also that the knowledge of the cyclic regular subgroups of the primitive permutation groups is necessary for the classification of polynomials by their monodromy groups, see for instance [19]. Of course knowing whether there is a regular subgroup in a permutation group is also important as, if X is a regular subgroup of (G,Ω), then for every $\omega\in\Omega$ the subgroup X is a complement in G to the stabilizer $A=G_\omega$.

Up to now the following special cases of the problem have been solved: The pairs (G,Ω) such that G contains a cyclic regular subgroup have been determined by W. Feit [14] (the insoluble ones), see also [16, Theorem 1.49], and by J.F. Ritt [34, p. 27] (the soluble ones)—see also G.A. Jones [23]. Those (G,Ω) such that G contains an abelian regular subgroup have been classified by C.H. Li, see [27].

Let us consider what has to be done to solve the general problem. Therefore, recall the subdivision of the primitive permutation groups into different types, see for instance [4,30]. Every primitive permutation group (G,Ω) of affine, of diagonal action, of twisted product action type or the product of groups of diagonal action type possesses a regular subgroup, see Proposition 1.1. Note that item (d) of this proposition is a consequence of Corollary 3 of [33].

Proposition 1.1. *Let (G,Ω) be a primitive permutation group. Then one of the following holds.*

(a) *G is almost simple*;
(b) *G is of diagonal type or the product of groups of diagonal type. In both cases a normal subgroup of the socle of G acts regularly on Ω*;
(c) *G is of affine type or twisted wreath type and the socle of G acts regularly on Ω*;
(d) *$G\leqslant H\wr \mathrm{S}_r$, $r\geqslant 2$, in product action on $\Omega=\Gamma^r$, where $\mathrm{soc}(G)=\mathrm{soc}(H)^r$, H is almost simple and primitive on Γ. If G has a regular subgroup, then (H,Γ) has a transitive core-free subgroup.*

The goal is to determine the (G,Ω) of type (a) or (d) which possess a regular subgroup X. In this paper, we focus on the case that G is almost simple.

We give a complete list of all the triples (G,Ω,X) with $\mathrm{soc}(G)$ an alternating, or a sporadic or an exceptional group of Lie type. Assume that G is a classical group. We present all the examples (G,Ω,X) which are known to us, see Tables 2–5 in Section 2, and we construct them all in Section 10.1. Our particular interest is in the 8-dimensional orthogonal groups of Witt index 4. This family of classical groups seems to be the most difficult one to analyze, as the group $\mathrm{P}\Omega_8^+(q)$ has lots of maximal subgroups, see [25], and admits lots of factorizations, see [31, Theorem A]. It also seems that most classical groups which admit a regular subgroup are of small dimension, see Section 10.1. In the case that $\mathrm{soc}(G)\cong \mathrm{P}\Omega_8^+(q)$, we also determine all the triples (G,Ω,X). In order to obtain all these triples, we also study the almost simple groups G with $G\cong \mathrm{P}\Omega_{2n+1}(q)$, see Theorems 4 and 5. Moreover, we characterize the Examples (a), (o), (p), (q) and (r) (see Section 10.1) in Theorems 7 and 8.

We intend to finish the full classification of classical groups with a regular subgroup by using induction on the dimension. In [6], we accomplish this goal for $\mathrm{soc}(G)$ a unitary group. I conjecture that the lists of examples given in Section 2 are complete. It seems to us that the groups of small dimension are the most difficult to analyze. The existence of a regular subgroup in a classical group seems to be a phenomenon which appears in groups of small dimension, but also with a small field of definition $\mathrm{GF}(q)$, see Tables 2–5. We found 11 examples with $q = 2$, 4 with $q = 3$, 5 with $q = 4$, 3 with $q = 8$, some with $q \in \{11, 23, 29, 59\}$ and two infinite families.

If X is a regular subgroup of G, then for every $\omega \in \Omega$, we have $G = AX$ and $A \cap X = 1$ where $A = G_\omega$, that is $G = AX$ is an *exact* factorization of G with factors A and X. It is interesting to note that in many of the examples for the classical groups one of the exact factors is the normalizer of a field extension, see Tables 2–5. Moreover, notice that for G a group of Lie type in characteristic p either $O_p(X) \neq 1$ or X is almost simple, apart from Examples (y), (u), (o) and (q).

Finally, note that Feit, Jones and Li used the classification of the finite simple groups—in this paper we also assume this.

An outline of the structure of the paper is as follows. In the next section, we present our results, that is Theorems 1–8. In Section 3, we introduce our notation concerning the classical groups, and in Section 4, we give our definition of an exceptional group of Lie type. Then in Section 5, we recall the definition of a Zsigmondy prime which will be essential in the discussion of the classical groups. We also study subgroups of classical groups of small dimension which are divisible by a Zsigmondy prime—these results will be needed in Sections 7–10. In Section 6, we present some simple facts about factorizations and discuss some obstacles which may occur. In Sections 7–10, we prove Theorems 1–3 and 4–8, respectively, and in Section 11, we give an application to the classification of primitive permutation groups with a regular subgroup. Finally, in the last section, we present more information about the history on primitive permutation groups with a regular subgroup and bring up Burnside-groups.

Let G be an almost simple group acting faithfully and primitively on a set Ω. As already noticed, if X is a regular subgroup, then it is a complement to the stabilizer G_ω of ω in Ω. Our method is to study closely the maximal factorizations of the almost simple groups given in [31]. Several times we use the following method to rule out (G, Ω) with G a group of Lie type in characteristic p. Assume there is a regular subgroup X. First we are able to prove that $O_p(X) \neq 1$. The theorem of Borel and Tits [17, 3.1.3] then yields that X is contained in a maximal parabolic subgroup of G. Last we show that this is not possible.

Shortly before submitting this paper the author learned from C.E. Praeger that she, M.W. Liebeck and J. Saxl would also be working on the classification of the primitive permutation groups with a regular subgroup.

In what follows for G a group we denote by $G^\#$ the set $G \setminus \{1\}$.

2. The results

The alternating and symmetric groups

Let us recall a special class of sharply 3-transitive groups: Let p be an odd prime and $e = 2m$ even. Then $M(p^e + 1)$ is the following group: Consider $K = \mathrm{PGL}_2(p^e)\langle\alpha\rangle$ with α a field automorphism of order 2. Then $K/\mathrm{L}_2(p^e)$ is elementary abelian of order 4 and therefore contains three subgroups of order 2. The group $M(p^e + 1)$ is the one which is neither $\mathrm{L}_2(p^e)\langle\alpha\rangle$ nor $\mathrm{PGL}_2(p^e)$. Notice that this group is called $M(p^e)$ in [22, p. 163].

Theorem 1. *Let (G,Ω) be a primitive permutation group with $\mathrm{soc}(G) = T \cong \mathrm{A}_n$, $n \geqslant 5$, and suppose that there is a subgroup X of G which acts regularly on Ω. Then one of the following holds, where ω is an element in Ω and $\Delta = \{1,\dots,n\}$. Conversely if (G,Ω) is a primitive permutation group satisfying one of the listed conditions, then G has a regular subgroup X.*

(a) $G = \mathrm{A}_n$.
  (a.a) $\Omega = \Delta$ and $G_\omega = \mathrm{A}_{n-1}$.
  (a.b) G_ω is sharply k-transitive on Δ and X is the pointwise stabilizer of a k-subset of Δ, for some $k \in \{3,4,5\}$, and one of the following holds.
    (a.b.a) $n = p^2+1$, with p a prime, $p \equiv 3 \bmod 4$, $k = 3$ and $G_\omega \cong M(p^2+1)$;
    (a.b.b) $n = 11$, $k = 4$ and $G_\omega \cong \mathrm{M}_{11}$;
    (a.b.c) $n = 12$, $k = 5$ and $G_\omega \cong \mathrm{M}_{12}$.
  (a.c) G_ω is k-homogeneous, but not k-transitive on Δ, for some $k \in \{2,3,4\}$, and one of the following holds. In the last two items p is a prime, $p \equiv 3 \bmod 4$, but $p \neq 7, 11, 23$.
    (a.c.a) $n = 9$, $k = 4$, $G_\omega \cong \mathrm{P\Gamma L}_2(8)$ and $X \cong \mathrm{S}_5$;
    (a.c.b) $n = 33$, $k = 4$, $G_\omega \cong \mathrm{P\Gamma L}_2(32)$ and $X \cong (\mathrm{A}_{29} \times \mathrm{A}_3) : 2$;
    (a.c.c) $n = p+1$, $k = 3$, $G_\omega \cong \mathrm{L}_2(p)$ and $X \cong \mathrm{S}_{p-2}$;
    (a.c.d) $n = p$, $k = 2$, $G_\omega \cong \mathrm{Frob}(p : (p-1)/2)$ and $X \cong \mathrm{S}_{p-2}$.
  (a.d) Ω is the set of k-subsets of Δ, for some $k \in \{2,3\}$, and one of the following holds. In the last item q is a prime power, $q \equiv 3 \bmod 4$.
    (a.d.a) $n = 8$, $k = 3$ and $X \cong \mathrm{AGL}_1(8)$;
    (a.d.b) $n = 32$, $k = 3$ and $X \cong \mathrm{A\Gamma L}_1(32)$;
    (a.d.c) $n = q$, $k = 2$ and $X \cong \mathrm{AGL}_1(q)/\langle -1 \rangle \cong \mathrm{Frob}(q : (q-1)/2)$.
  (a.e) $n = 8$, $G_\omega \cong 2^3 : \mathrm{L}_3(2)$, $|\Omega| = 15$ and $X \cong \mathbb{Z}_{15}$.

(b) $G = \mathrm{S}_n$.
  (b.a) $G_\omega \cap \mathrm{A}_n$ is a subgroup of index 2 in G_ω and is as in (a.a), (a.b.a), (a.c.c), (a.c.d) or (a.d).
  (b.b) G_ω is sharply k-transitive on Δ, for some $k \in \{2,3\}$, $X \cong \mathrm{S}_{n-k}$ and one of the following holds. In both cases p is a prime, $p \equiv 3 \bmod 4$.
    (b.b.a) $n = p+1$, $k = 3$ and $G_\omega \cong \mathrm{PGL}_2(p)$;
    (b.b.b) $n = p$, $k = 2$ and $G_\omega \cong \mathrm{Frob}(p : (p-1))$.
  (b.c) $n = 6$, $G_\omega \cong \mathrm{PGL}_2(5)$ is transitive on Δ and X is a subgroup of G of order 6.

(c) $T = \mathrm{A}_6$. G contains a subgroup isomorphic to $\mathrm{PGL}_2(9)$, G_ω is the normalizer of a Sylow 3-subgroup of G and $X \cong \mathbb{Z}_{10}$, or D_{10}.

Remark. If (G,Ω) is as in (b.b.a) or (b.b.b), but $p \neq 7, 11, 23$, then $(\mathrm{A}_n, \Omega, \mathrm{A}_n \cap G_\omega)$ is as in item (a.c.c) or (a.c.d) of the theorem, respectively.

The sporadic groups

Theorem 2. *Let (G,Ω) be a primitive permutation group with $\mathrm{soc}(G) = T$ a sporadic simple group. Suppose that there is a subgroup X of G which acts regularly on Ω. Then (G, A, X) are as follows, where A is the stabilizer in G of an element in Ω. Conversely if (G,Ω) is a permutation group with subgroup X as given in Table 1, then G has a regular subgroup which is isomorphic to X.*

Table 1
Regular subgroups of the sporadic groups

No	T	G	$A \cap T$	Ω	X
(a.a)	M_{11}	$T = \mathrm{Aut}(T)$	M_{10}	Set of points of Steiner system $\mathcal{S} = S(4,5,11)$ related to T	$\mathbb{Z}_{11}$
(a.b)		T	$M_9.2 \cong 3^2 : SD_{16}$	Set of duads of Steiner system $\mathcal{S}$	$\mathrm{Frob}(11:5)$
(b.a)	M_{12}	T	M_{11}	Set of points of Steiner system $S(4,5,11)$ related to T	$2^2 \times 3, A_4, 2 \times S_3$
(b.b)		T	$L_2(11)$		$3^2 : SD_{16}$
(c)	M_{22}	$\mathrm{Aut}(T)$	M_{21}	Set of points of Steiner system $S(4,6,22)$ related to T	$\mathrm{Frob}(11:2)$
(d.a)	M_{23}	$T = \mathrm{Aut}(T)$	M_{22}	Set of points of Steiner system $\mathcal{S} = S(4,7,23)$ related to T	$\mathbb{Z}_{23}$
(d.b)		T	$M_{21} : 2$	Set of duads of $\mathcal{S}$	$\mathrm{Frob}(23:11)$
(d.c)		T	$\mathrm{Frob}(23:11)$		$M_{21} : 2, 2^4 : A_7$
(d.d)		T	$2^4 : A_7$	Set of blocks of $\mathcal{S}$	$\mathrm{Frob}(23:11)$
(e.a)	M_{24}	$T = \mathrm{Aut}(T)$	M_{23}	Set of points of Steiner system $S(5,8,24)$ related to T	$D_8 \times 3, (2^2 \times 3) : 2,$ $S_4, D_{24}, 2 \times A_4$
(e.b)			$L_2(23)$		$2^4 : A_7, M_{21} : 2$
(f)	J_2	$\mathrm{Aut}(T)$	$U_3(3)$	Set of points of the rank three graph for T	$X \nleqslant T$, $X \cong 5^2 : 4$ each eigenvalue of $X/O_5(X)$ on $O_5(X)$ generates GF(5)[#]
(g)	HS	$\mathrm{Aut}(T)$	M_{22}	Set of points of the Higman–Sims graph related to T	$X \nleqslant T$, $X \cong 5^2 : 4$ the eigenvalues of $X/O_5(X)$ on $O_5(X)$ are $1, w; w, w$ or w, w^{-1} with $\langle w \rangle =$ GF(5)[#]
(h)	He	$\mathrm{Aut}(T)$	$Sp_4(4) : 2$	Set of points of the rank five graph for T	$7^{1+2} : \mathbb{Z}_6$

In particular, the rank 3 graphs for J_2 and HS, respectively, as well as the rank 5 graph for He are Cayley graphs.

The exceptional groups of Lie type

If G is an exceptional group of Lie type, then there is no example:

Theorem 3. *Let (G, Ω) be a primitive permutation group with* $\mathrm{soc}(G) = T$ *an exceptional group of Lie type. Then there is no regular subgroup in G.*

The classical groups

Let G be a classical group. In Tables 2–5 we present all the examples which are known to us and in Section 10.1 we construct these examples. All the tables are organized as follows. In the first column we list the name of the example, in the second the socle T of the almost simple group and in the third a group G such that $T \leqslant G \leqslant \mathrm{Aut}(T)$, such that G acts primitively on Ω

Table 2
Examples with T a linear group

No	T	G	T_ω	Ω	X
(w)	$L_2(11)$	T	A_5		$\mathbb{Z}_{11}$
(x)	$L_2(11)$	$PGL_2(11)$	A_4		$\mathrm{Frob}(11:5)$
(x)	$L_2(23)$	T	S_4		$\mathrm{Frob}(23:11)$
(w)	$L_2(29)$	T	A_5		$\mathrm{Frob}(29:7)$
(w)	$L_2(59)$	T	A_5		$\mathrm{Frob}(59:29)$
(v)	$L_3(3)$	T	$\mathrm{Frob}(13:3)$		$3^2:2{}^{\cdot}D_8$
(j)	$L_3(4)$	$P\Gamma L_3(4)$	$\mathrm{Frob}(7:3)$		$2^4:(3\times D_{10}){}^{\cdot}2$
((d))	$L_4(2)$	$\mathrm{Aut}(T)$	$L_3(2)$	$V_1\oplus V_3$	$P\Gamma L_2(4)$
((k))	$L_4(2)$	T	$\Gamma L_2(4)$		$2^3:7$
(s)	$L_4(3)$	$PGL_4(3)$	$(4\times L_2(9)):2$		$3^3:(13:3\times 2)$
(k)	$L_4(4)$	$T:2$	$(5\times L_2(16)).2$		$2^6:(7:3\times 3^2):2$
(y)	$L_5(2)$	T	$\mathrm{Frob}(31:5)$		$2^6:(S_3\times L_3(2))$
(y)	$L_5(2)$	T	$2^6:(S_3\times L_3(2))$	lns/pls	$\mathrm{Frob}(31:5)$
(v)	$L_2(q)$ $q\equiv 3(4)$	$PGL_2(q)$, $q=7$ T, $q\neq 7$	D_{q+1}		$\mathrm{Frob}(q:(q-1)/2)$
(u)	$L_n(q)$	$PGL_n(q)$	$q^{n-1}:\frac{GL_{n-1}(q)}{(n,q-1)}$	pts/hyperpls	$\mathbb{Z}_{\frac{(q^n-1)}{(q-1)}}$, more

Table 3
Examples with T a symplectic group

No	T	G	T_ω	Ω	X
((b))	$Sp_4(2)'$	$Sp_4(2)$	$L_2(4)$	$\{1,\dots,6\}$	$\mathbb{Z}_6$, S_3
(l)	$Sp_4(4)$	$T:2$	$L_2(16):2$		$P\Gamma L_2(4)$
(e)	$Sp_6(2)$	T	$G_2(2)$		$P\Gamma L_2(4)$
(h)	$Sp_6(2)$	T	$L_2(8):3$		$2^4.L_2(4)$
(t)	$PSp_6(3)$	$T:2$	$L_2(27):3$		$3^{1+4}:2^{1+4}:(5:4)$
(m)	$Sp_6(4)$	$T:2$	$G_2(4)$		$P\Gamma L_2(16)$
(g)	$Sp_8(2)$	T	$SO_8^-(2)$		$P\Gamma L_2(4)$

Table 4
Examples with T an orthogonal group

No	T	G	T_ω	Ω	X
(f)	$\Omega_8^+(2)$	T	$Sp_6(2)$	non-sing. pts	$P\Gamma L_2(4)$
(i)	$\Omega_8^+(2)$	T	A_9		$2^6:\mathbb{Z}_{15}$, $2^4.L_2(4)$, $3:(2^4:(5:4))$
(n)	$\Omega_8^+(4)$	$T:2$	$Sp_6(4)$	non-sing. pts	$P\Gamma L_2(16)$

Table 5
Examples with T a unitary group

No	T	G	T_ω	Ω	X
(a)	$U_4(2)$	T	$2^4:A_5$	tot. iso. lns	3_+^{1+2}, 3_-^{1+2}
(o)	$U_3(8)$	$T:3^2$	$2^{3+6}:21$	tot. iso. pts	$19:9\times 3$
(p)	$U_3(8)$	$T:3^2$	$19:9\times 3$		$2^{3+6}:21$
(r)	$U_4(3)$	$T:2$	$L_3(4)$		$3^4:2$, $O_3(X):2$
(q)	$U_4(8)$	$T:3$	$2^{12}:7.L_2(64)$	tot. iso. lns	$9{}^{\cdot}(19:9\times 3)$

and such that G contains a regular subgroup. In all examples except in the families of examples listed in (u) G is the smallest group satisfying these properties. The fourth column contains the structure of the stabilizer T_ω, for an $\omega \in \Omega$. In all the cases Ω can be considered as the set of cosets of $N_G(T_\omega)$ in G. If Ω is a nice object, then we list it in the fifth column. These objects, except in row (b), are always contained in the natural module for T. In row (d) we mean by $V_1 \oplus V_3$ the set of antiflags consisting of a 1 and a 3-dimensional subspace which intersect trivially. In the last column we list the isomorphism types of all the regular subgroups for the given G and Ω.

We first prove the following theorem in order to be able to study $\mathrm{P}\Omega_8^+(q)$.

Theorem 4. *Let (G, Ω) be a primitive permutation group with $T = \mathrm{soc}(G) \cong \mathrm{P}\Omega_7(q)$, q even or odd. Suppose that $G_\omega \cap T \cong \mathrm{G}_2(q)$, for ω in Ω. Then G has a regular subgroup X if and only if $q \in \{2, 4\}$. If there is a regular subgroup X in G, then $X \cong \mathrm{P\Gamma L}_2(q^2)$.*

The next theorem, which is a negative statement, we only need for $n = 3$, but prove in full generality.

Theorem 5. *Let (G, Ω) be a primitive permutation group with $T = \mathrm{soc}(G) \cong \mathrm{P}\Omega_{2n+1}(q)$, q odd. Let V be the natural module for T and assume that Ω equals the set of totally singular subspaces of dimension i of V, for some i in $\{1, \dots, n\}$. Then there is no regular subgroup in G.*

Then we are showing the main result of this section.

Theorem 6. *Let (G, Ω) be a primitive permutation group with $T = \mathrm{soc}(G) \cong \mathrm{P}\Omega_8^+(q)$. Suppose there is a subgroup X of G which acts regularly on Ω. Then (G, Ω, X) is one of the examples* (f), (i) *or* (n) *listed in Section* 10.1, *see also Table* 4.

In order to characterize the examples for the unitary groups in their action on the set of totally isotropic i-subspaces of their natural module we show the following:

Theorem 7. *Let $T \cong \mathrm{U}_n(q), n \geqslant 3$, and let $T \leqslant G \leqslant \mathrm{Aut}(T)$. Suppose there is a subgroup X of G which acts regularly on the set Ω of totally isotropic subspaces of dimension i of the natural T-module V, for some $1 \leqslant i \leqslant n/2$ if n is even or $1 \leqslant i \leqslant (n-1)/2$ if n is odd. Then $(n, q) =$* (4, 2), (3, 8) *or* (4, 8) *and the pairs (Ω, X) are as in Examples* (a), (o) *or* (q) *in Section* 10.1, *see also Table* 5.

Finally we also characterize Examples (p) and (r).

Theorem 8. *Let (G, Ω) be a primitive permutation group with $T = \mathrm{soc}(G) \cong \mathrm{U}_n(q), n \geqslant 3$. Suppose the following*:

(a) *Ω is not the set of non-singular* 1*-spaces of the natural T-module*;
(b) *There is a subgroup X of G which acts regularly on the set Ω.*

Then (T, Ω, X) are as in Examples (a), (o), (p), (r) *or* (q) *in Section* 10.1, *see also Table* 5.

Remarks. (a) In case (u) there are more regular subgroups; for instance if $(n, q) = (2, 11)$ or $(2, 23)$ then there is also a regular subgroup isomorphic to A_4 or S_4, respectively.

(b) In cases (s) and (k), the subgroup X is the full 1-dimensional affine group.

(c) In the tables we assume $T \not\cong \mathrm{A}_5, \mathrm{A}_6, \mathrm{A}_8$. But as they are part of a family of examples we included Examples (d) and (k) for $\mathrm{A}_8 \cong \mathrm{L}_4(2)$ and (b) for $\mathrm{A}_6 \cong \mathrm{Sp}_4(2)'$.

Remarks. (a) There is a series of examples over GF(2): (b)-(c)-(d)-(e)-(f)-(g) with

$$\mathrm{S}_6 < \mathrm{S}_7 < \mathrm{L}_4(2):2 < \mathrm{Sp}_6(2) < \Omega_8^+(2) < \mathrm{Sp}_8(2).$$

(b) The part (d)-(e)-(f) of the latter series is analogous to the GF(4)-series (np)-(m)-(n) with

$$\mathrm{L}_4(4):2 < \mathrm{Sp}_6(4) < \Omega_8^+(4),$$

where (np) is non-primitive (see Section 10.1).

(c) Example (b) for $G = \mathrm{S}_6 \cong \mathrm{Sp}_4(2)$ has no GF(4)-analogue, see the Atlas [11, p. 44], and the series cannot be extended to $\Omega_{10}^+(2)$, $\mathrm{Sp}_8(4)$, respectively (for $q = 2$ see [11, p. 146]). Examples (d) and (np) do not exist for a group G with $T = \mathrm{soc}(G) \cong \mathrm{L}_4(q)$ and $q \neq 2, 4$ for arithmetical reasons, and Examples (d) and (np) cannot be constructed for $\mathrm{L}_n(q)$ with q in $\{2, 4\}$ and with $n > 4$ even, either.

(d) P. Cameron and W.M. Kantor [9, Theorems II and III] have classified the transitive collineation groups which are transitive on antiflags. One case is missing in their classification as has been observed in [31, (3.1.2)]; namely, the case $q = 4$ and $\mathrm{L}_2(16) \trianglelefteq G$ should appear in the statement of Theorem III. Their classification yields that if T is a linear group and Ω the set of antiflags, then we obtain Example (d) or Example (np), precisely. (It also has been pointed out by Shreeram Abhyankar that some proofs in [9] are not complete [Personnal Communication with G. Stroth, 2003].)

3. The classical groups

Let V be a vector space of dimension n over a finite field k, and let $(\,,\,)$ denote a sesquilinear form from $V \times V$ to k. We distinguish four cases:

Case L. $(\,,\,)$ is trivial, that is $(u, v) = 0$ for all u, v in V.

Case Sp. $(\,,\,)$ is a non-singular skew-symmetric bilinear form, that is $(u, v) = -(v, u)$ and $(u, u) = 0$ for all u, v in V.

Case O. $(\,,\,)$ is a non-singular symmetric bilinear form, and there is a quadratic form Q from V to k with $(\,,\,)$ as its associated bilinear form.

Case U. $(\,,\,)$ is a non-singular sesquilinear form, that is $(\,,\,)$ is left linear, the field $k = \mathrm{GF}(q^2)$ possesses an involutory automorphism and

$$(v, u) = (u, v)^q \quad \text{for all } u, v \in V.$$

When n is even, the Case O splits into two subcases:

Case O$^+$. Occurs if Q is a quadratic form of $+$ type.

Case O^-. Occurs if Q is a quadratic form of $-$ type.

In Cases L, Sp, O^+, O^-, O, U the respective groups of isometries having determinant 1 are called

$$\mathrm{L}_n(k),\quad \mathrm{Sp}_n(k),\quad \mathrm{SO}_n^+(k),\quad \mathrm{SO}_n^-(k),\quad \mathrm{SO}_n(k),\quad \mathrm{SU}_n(k).$$

In general, the groups $\mathrm{PSO}_n^+(k)$, $\mathrm{PSO}_n^-(k)$ and $\mathrm{PSO}_n(k)$ (these groups are obtained from the groups of isometries by factoring out the center) are not simple, but have a subgroup of index at most 2, which we denote by $\mathrm{P}\Omega_n^+(k)$, $\mathrm{P}\Omega_n^-(k)$ and $\mathrm{P}\Omega_n(k)$, respectively. The classical simple groups are $\mathrm{PSL}_n(k) = \mathrm{L}_n(k)$, $\mathrm{PSp}_n(k)$, $\mathrm{P}\Omega_n^+(k)$, $\mathrm{P}\Omega_n(k)$, $\mathrm{PSU}_n(k) = \mathrm{U}_n(q)$.

Now we explain the symbol "ˆ" which appears to the left of several subgroups in Sections 7, 8, 9 and 10 and which is taken from [31]. Let $\widehat{T}$ be a classical linear group on V with center Z (so that $T = \widehat{T}/Z$ is a classical simple group), and let $\widehat{G}$ be a group such that $\widehat{T} \trianglelefteq \widehat{G} \leqslant \mathrm{GL}(V)$. For a subgroup U of $\widehat{G}$, we denote by $\hat{}U$ the subgroup $(UZ \cap \widehat{T})/Z$ of T.

Standard bases

We call the k-vector space V which is used to construct the classical groups $G = \mathrm{Sp}_n(q), \mathrm{GO}_n^\varepsilon(q)$ and $\mathrm{GU}_n(q)$ the *natural module* for G, as well as for T, where $T = G/Z(G)$. In each of the cases Sp, U and O, the vector space has particularly convenient bases, called *standard* bases, which we describe now.

Case Sp. Here the dimension is $n = 2m$, and there is a basis $\{e_1, \dots, e_m, f_1, \dots, f_m\}$ such that

$$(e_i, e_j) = (f_i, f_j) = 0, \qquad (e_i, f_j) = \delta_{ij} \quad \text{for every } i, j.$$

Case U. If n is even, say $n = 2m$, there is a basis $\{e_1, \dots, e_m, f_1, \dots, f_m\}$ such that

$$(e_i, e_j) = (f_i, f_j) = 0, \qquad (e_i, f_j) = \delta_{ij} \quad \text{for every } i, j.$$

If $n = 2m + 1$, there is a basis $\{e_1, \dots, e_m, f_1, \dots, f_m, d\}$ satisfying the additional conditions that

$$(e_i, d) = (f_i, d) = 0, \qquad (d, d) = 1.$$

Case O. In Case O^+ we have $n = 2m$ and a basis $\{e_1, \dots, e_m, f_1, \dots, f_m\}$ such that

$$Q(e_i) = Q(f_i) = (e_i, e_j) = (f_i, f_j) = 0, \qquad (e_i, f_j) = \delta_{ij} \quad \text{for every } i, j.$$

In Case O^- we have $n = 2m$ and a basis $\{e_1, \dots, e_{m-1}, f_1, \dots, f_{m-1}, d_1, d_2\}$ with e_i, f_i behaving as above, and we may take

$$Q(d_1) = (d_1, d_2) = 1, \qquad Q(d_2) = \sigma, \qquad (e_i, d_j) = (f_i, d_j) = 0 \quad \text{for every } i, j,$$

where the quadratic equation $x^2 + cx + \sigma$ is irreducible over k with $c = 1$ if $\operatorname{char} k = 2$ and $c = 2$ else.

If $n = 2m + 1$, then q is odd and there is a basis $\{e_1, \ldots, e_m, f_1, \ldots, f_m, d\}$ with e_i, f_i behaving as above, and

$$(e_i, d) = (f_i, d) = 0, \qquad Q(d) = 1.$$

4. The exceptional groups of Lie type

We call all the simple groups of Lie type which are not classical exceptional. So they are

$$\mathrm{G}_2(q),\quad q > 2, \qquad \mathrm{F}_4(q), \qquad \mathrm{E}_6(q), \qquad \mathrm{E}_7(q), \qquad \mathrm{E}_8(q),$$
$$^2\mathrm{B}_2(q),\quad q = 2^{2n+1} > 2, \qquad ^2\mathrm{G}_2(q),\quad q = 3^{2n+1} > 3, \qquad ^2\mathrm{F}_4(q),\quad q = 2^{2n+1}$$
$$\text{or} \quad ^3\mathrm{D}_4(q), \qquad ^2\mathrm{E}_6(q).$$

5. On certain prime divisors of classical groups

Recall that Zsigmondy [41] has shown the following.

Lemma 5.1. [41] *Let p be a prime and $s \in \mathbb{N}$. Then one of the following holds*:

(a) *there exists a prime r (called a* Zsigmondy prime for $p^s - 1$) *which divides $p^s - 1$, but does not divide $p^i - 1$, for $i = 1, \ldots, s - 1$*;
(b) *$s = 2$ and p is a Mersenne prime*;
(c) *$s = 6$ and $p = 2$.*

Notice, if r is a Zsigmondy prime for $p^s - 1$, then $r \equiv 1 \bmod s$.

Let $q = p^a$ for some prime p. If p^n is not as in statements (b) or (c) of Lemma 5.1, then we denote by q_n the largest Zsigmondy prime for $q^n - 1$. Notice that there is a paper by R.M. Guralnick, T. Pentilla, C.E. Praeger, J. Saxl where the authors determine all the maximal subgroups of the classical groups for which the orders are divisible by certain Zsigmondy primes [18].

Let $t = xp^i$ be a natural number with $(x, p) = 1$. We denote the p-part p^i of t by t_p. Frequently we will use the following fact.

Lemma 5.2. *For p a prime, the p-part $|\mathrm{S}_n|_p$ of $|\mathrm{S}_n|$ is at most $p^{(n-1)/(p-1)}$.*

According to 5.1 the Zsigmondy prime q_4 always exists.

Lemma 5.3. *Let $T \cong \mathrm{L}_4(q)$, let $T \leqslant G \leqslant \mathrm{Aut}(T)$ and let M be a maximal subgroup of G which is divisible by q_4. Then $T \cap M$ is isomorphic to one of the following groups*:

(a) $(\mathrm{L}_2(q^2) \times (q + 1)).2$;
(b) $\mathrm{PSp}_4(q)$ *and* $q \equiv 1(4)$ *or* $\mathrm{PSp}_4(q).(2, q - 1)$ *and* $q \equiv 0, 2, 3(4)$;
(c) $\mathrm{P}\Omega_4^-(q).2$, *and q odd*;
(d) $2^4.\mathrm{A}_6$ *and* $q = p \equiv 5(8)$ *or* $2^4.\mathrm{S}_6$ *and* $q = p \equiv 1(8)$;
(e) A_7 *and* $q = p \equiv 1, 2, 4(7)$ *and* $q_4 = 5$;
(f) $\mathrm{U}_4(2)$ *and* $q = p \equiv 1(6)$ *and* $q_4 = 5$.

Proof. We obtain this list of subgroups immediately by checking the list of maximal subgroups of $\mathrm{L}_4(q)$ given in [24], see also [18]. $\square$

Later we will need the following result:

Corollary 5.4. *Let $T \cong \mathrm{L}_4(q)$, let $T \leqslant G \leqslant \mathrm{Aut}(T)$ and let M be a maximal subgroup of G which is divisible by the odd part u of $(q+1)(q^2+1)/t$ with $t = ((q+1)(q^2+1), e)$ where $q = p^e$. Then $T \cap M$ is isomorphic to one of the groups in* (a)–(c) *of* 5.3.

Proof. Assume that $T \cap M$ is isomorphic to one of the groups in (d)–(f) of 5.3. Assume (d). Then $(p+1)_2 = 2$ and $p^2 + 1 \equiv 1(8)$. As in this case u divides $3^2 \cdot 5$, it follows that 3 divides $p+1$ and $p^2 + 1 = 2 \cdot 5$. Thus $p = 3$, a contradiction.

Now assume (e). Again u divides $3^2 \cdot 5$. If 4 divides $p+1$, then $p^2 + 1 = 10, 30$ or 90, which is not possible. Hence $(p+1)_2 = 2$ and $p + 1 = 6$ or 18, which yields a contradiction in both cases.

Thus (f) holds. Then $p+1$ and p^2+1 are not divisible by 3, which implies $p^2 + 1 = 10$ and $p = 3$, a contradiction. This shows the assertion. $\square$

Lemma 5.5. *Let $T \cong \mathrm{PSp}_4(q)$, let $T \leqslant G \leqslant \mathrm{Aut}(T)$ and let M be a maximal subgroup of G which is divisible by the Zsigmondy prime q_4. Then $T \cap M$ is isomorphic to one of the following groups*:

(a) $\mathrm{PSp}_2(q^2).2 \cong \mathrm{L}_2(q^2).2$;
(b) $2^4.\Omega_4^-(2) \cong 2^4.\mathrm{A}_5$ *and* $q = p \equiv \pm 3(8)$ *or* $2^4.\mathrm{O}_4^-(2)$ *and* $q = p \equiv \pm 1(8)$;
(c) $\mathrm{Sz}(q)$ *and q even*;
(d) A_6 *and* $q = p \equiv 2, \pm 5(12)$ *or* S_6 *and* $q = p \equiv \pm 1(12)$;
(e) A_7 *and* $q = 7$.

Proof. Again, we obtain this list of subgroups directly by checking the list of maximal subgroups of $\mathrm{PSp}_4(q)$ given in [24], see [18] as well. $\square$

Lemma 5.6. *Let $T \cong \mathrm{U}_4(q)$, let $T \leqslant G \leqslant \mathrm{Aut}(T)$ and let M be a maximal subgroup of G which is divisible by the Zsigmondy prime q_4. Then $T \cap M$ is isomorphic to one of the following groups*:

(a) $\hat{}(q^4 : \mathrm{SL}_2(q^2).2.\mathbb{Z}_{q-1})$;
(b) $\hat{}(\mathrm{SL}_2(q^2).\mathbb{Z}_{q-1}.2)$ *and* $q \geqslant 4$;
(c) $\mathrm{PSp}_4(q).2$ *and* $q \equiv 1(4)$ *or* $\mathrm{PSp}_4(q)$ *and* $q \equiv 0, 2, 3(4)$;
(d) $\Omega_4^-(q).2 \cong \mathrm{L}_2(q^2).2$*, and q odd*;
(e) $2^4.\mathrm{S}_6$ *and* $q = p \equiv 7(8)$ *or* $2^4.\mathrm{A}_6$ *and* $q = p \equiv 3(8)$;
(f) A_7 *and* $q = p \equiv 3, 5, 6(7)$;
(g) $\mathrm{U}_4(2)$ *and* $q = p \equiv 5(6)$;
(h) $\mathrm{L}_3(4)$ *and* $q = 3$.

Proof. Again, we obtain this list of subgroups directly by checking the list of maximal subgroups of $\mathrm{U}_4(q)$ given in [24], see [18] as well. $\square$

Lemma 5.7. *Let T be a simple group which is isomorphic to $\mathrm{L}_4(q)$, $\mathrm{U}_4(q)$ or $\mathrm{Sp}_4(q)$ and let $T \leqslant G \leqslant \mathrm{Aut}(T)$. There is a subgroup in G whose order is divisible by $q^3 q_4$, with q_4 the Zsigmondy prime for q^4-1, and whose order divides $q^3(q^4-1)/(2,q-1)$ if and only if $q = 2^{2^i}$ with $0 \leqslant i \leqslant 2$.*

Proof. First let $q=2$. In $T \cong \mathrm{L}_4(2) \cong \mathrm{A}_8$ there exists a subgroup U isomorphic to S_5. As $|U| = 2^3(2^4-1)$, this subgroup satisfies the requirements on the order of the lemma. Now let $q=4$ and $G \cong \mathrm{L}_4(4):2$ where the outer involution acts on $G' = T$ as a graph automorphism. Then there is subgroup $M \cong \mathrm{Sp}_4(4) \times 2$ in G, which contains a subgroup $U \cong \mathrm{O}_4^-(4) \times 2 \cong \mathrm{L}_2(16):2 \times 2$. Here $|U| = 4^3(4^4-1)$. Finally, let $q=16$ and $G \cong \mathrm{Aut}(\mathrm{L}_4(16)) \cong \mathrm{L}_4(16):(2 \times 4)$. Let g be a graph automorphism of G. Then $C_G(g) \cong \mathrm{Sp}_4(16):4 \times 2$. In $C_G(g)$ we find a subgroup isomorphic to $\mathrm{L}_2(2^8):8 \times 2$, which proves one direction of the asserted equivalence.

For the rest of the proof let $q \neq 2, 4, 16$ and assume that there exists a subgroup X of G such that $|X|$ is divisible by $q^3 q_4$ and such that $|X|$ divides $q^3(q^4-1)/(2,q-1)$.

Assume that $T \cong \mathrm{Sp}_4(q)$ or $\mathrm{L}_4(q)$. Then $T \cap X$ is contained in one of the groups listed in 5.5 or in 5.3. The condition that q^3 has to divide $|X|$, as well, and that $|X|$ divides $q^3(q^4-1)/(2,q-1)$ gives a contradiction in all cases.

Thus $T \cong \mathrm{U}_4(q)$. Then $T \cap X$ is contained in one of the groups listed in Lemma 5.6. The condition on the order of $T \cap X$ yields that X is a subgroup of a maximal subgroup M listed in (a) or (h). If M is of type (a), then $O_p(M)$ is an irreducible module for $M/O_p(M)$, where q is a power of the prime p. Therefore, M does not contain a subgroup of order $|X|$ in that case. By checking the list of subgroups of $\mathrm{L}_3(4)$ in the Atlas [11], we see that M cannot be of type (h), as well. Thus there is no counterexample to our lemma. □

Table 6 is taken from [33, Table 10.1]. The first column gives the simple groups T to which this table applies, apart from the exceptions which are indicated in the last column. The third column gives certain sets Π of prime divisors of $|T|$ under the conditions given by column 2, and the fourth column lists those subgroups A of T which have order divisible by all the primes in Π.

Table 6
Subgroups whose orders are divisible by certain primes

T	Conditions	Set Π of primes dividing $\|A\|$	Possible A	Exceptions
$\mathrm{L}_n(q)$	n even	q_n, q_{n-1}	none	$\mathrm{L}_6(2)$
$n \geqslant 5$	n odd	q_n, q_{n-1}, q_{n-2}	none	$\mathrm{L}_7(2)$
$\mathrm{U}_n(q)$	n odd	q_{2n}, q_{2n-4}	none	$\mathrm{U}_5(2)$
$n \geqslant 5$	$n \equiv 0 \bmod 4$	q_{2n-2}, q_n	none	
	$n \equiv 2 \bmod 4$	$q_{2n-2}, q_{2n-6}, q_{n/2}$	none	$\mathrm{U}_6(2)$
$\mathrm{PSp}_{2m}(q)$	m odd	q_{2m}, q_{2m-2}, q_m	none	$\mathrm{Sp}_6(2)$
($m \geqslant 3$) or	m even	$q_{2m}, q_{2m-2}, q_{2m-4}$	$M \trianglerighteq \Omega_{2m}^-(q)$	$\mathrm{Sp}_8(2)$
$\mathrm{P}\Omega_{2m+1}(q)$				
($m \geqslant 3$)				
$\mathrm{P}\Omega_{2m}^-(q)$		$q_{2m}, q_{2m-2}, q_{2m-4}$	none	$\Omega_8^-(2)$
$m \geqslant 4$				$\Omega_{10}^-(2)$
$\mathrm{P}\Omega_{2m}^+(q)$	m odd	q_m, q_{2m-2}, q_{2m-4}	none	$\Omega_{10}^+(2)$
$m \geqslant 4$	m even	$q_{m-1}, q_{2m-2}, q_{2m-4}$	$M \trianglerighteq \Omega_{2m-1}(q)$	$\Omega_8^+(2)$

6. Simple facts about factorizations

In this section we will present some simple facts about factorizations which we shall use throughout the text. In order to determine the primitive groups admitting a regular subgroup we shall quote the classification of the maximal factorizations of the almost simple groups by Liebeck, Praeger and Saxl [31] many times. In this section we also analyze for which primitive permutation groups we are able to quote [31]. In all what follows G denotes a finite group.

Lemma 6.1. *Let A and B two subgroups of G. The following statements are equivalent*:

(a) $G = AB$.
(b) $G = AB^g$ *for every* $g \in G$.
(c) $G = AB^n$ *for every* $n \in N_{\mathrm{Aut}(G)}(A)$.
(d) $|G : A| = |B : A \cap B|$.
(e) $|G : B| = |A : A \cap B|$.

Proof. The equivalence of (a), (c), (d) and (e) is obvious. Assume (a). Then B acts transitively on the cosets of A in G, so B^g acts transitively on these cosets, as well. Thus (b) holds. Clearly, (b) implies (a). $\square$

We will mainly use the following lemma in the proof of Theorem 2.

Lemma 6.2. *If A and X are subgroups of G, then X is a complement to A if and only if $|G| = |A||X|$ and no G-conjugacy class is represented in both $X^\#$ and $A^\#$.*

Proof. Assume there is an element $x \in X$ and an element $g \in G$ such that $x^g \in A$. Then A and X^g intersect non-trivially, which implies $G \neq AX^g$ in contradiction with Lemma 6.1. $\square$

The next three lemmas, in particular 6.3 will be applied in Sections 7, 8, 9 and 10.

Lemma 6.3. *Let X be a complement to the subgroup A in G and let B be a maximal subgroup of G containing X. Then $G = AB$ and X is a complement to $A \cap B$ in B. In particular, if B_1 and B_2 are maximal subgroups of B containing $B \cap A$ and X respectively, then $B = B_1 B_2$, moreover, $|B : B_2|$ divides $|B : X| = |B \cap A|$ and $|B : B_1|$ divides $|B : A \cap B| = |X|$.*

Lemma 6.4. *Let the group $G = U \times N$ be a direct product of two of its subgroups and suppose that $G = AB$ for two subgroups A and B of G which do not project trivially or onto N. Then $N = \overline{A}\overline{B}$ with $\overline{A}$ and $\overline{B}$ the projections of A and B on N. In particular, if N is almost simple and $E(N)$ is not contained in $\overline{A}$ or $\overline{B}$, then we obtain a core-free factorization of N.*

Lemma 6.5. *Let T be a simple classical group defined over the field K with natural module V, let $T \leqslant G \leqslant \mathrm{Aut}(T)$ and suppose that $G = AB$ with $A = G_W$ for a subspace W of V and B the normalizer of a field extension $K \subseteq L$. Then $A \cap B$ is a subgroup of the stabilizer of the subspace $W_L := \langle W \rangle_L$ of V_L, that is V considered as an L-vector space. Let $(\,,)$ be a form on V_L which is left invariant by B. Then $(\,,)_{\mathrm{trace}} = \sum_{\sigma \in \mathrm{Gal}(K \subseteq L)} (\,,)^\sigma$ is a form on V and if W is non-singular with respect to $(\,,)_{\mathrm{trace}}$, then W_L is non-singular with respect to $(\,,)$.*

Last we discuss some possible obstacles. Let G be an almost simple group with socle T. M.W. Liebeck, C.E. Praeger and J. Saxl determined all the factorizations $G = AB$ of the group G such that A and B are maximal core-free subgroups of G. Now suppose that G acts primitively on a set Ω and that X is a subgroup of G which is regular on Ω. Then $G = AX$ for $A = G_x$ the stabilizer in G of an element $x \in \Omega$. Let B be a subgroup of G containing X which is maximal with respect being core-free. Then B is not necessarily a maximal subgroup of G.

For a subgroup A of G, write

$$A \max^- G$$

to mean that A is maximal among core-free subgroups of G. And write

$$A \max^+ G$$

to mean that A is both core-free and maximal in G.

If $G = AB$ with $A, B \max^- G$, call the factorization a $\max^-$ factorization of G. If $G = AB$ with $A, B \max^+ G$, call the factorization a $\max^+$ factorization of G. This notion was introduced in [32] by Liebeck, Praeger and Saxl. We also call a $\max^+$ factorization of G simply a *maximal factorization* of G. In [32], the following has also been proven.

Lemma 6.6. [32, Lemma 2] *Suppose that $G = AB$ with core-free subgroups of G. Let $G^\star = AT \cap BT$, $A^\star = A \cap G^\star$, and $B^\star = B \cap G^\star$. Then*

(a) *$G^\star = A^\star B^\star$ and $A^\star T = B^\star T = G^\star$;*
(b) *either $A^\star \max^+ G^\star$ or $A \cap T$ is non-maximal in T; similarly for $B^\star$.*

Lemma 6.7. [32, Corollary 3] *If $G = AB$ is a core-free factorization, then $G^\star$ has a $\max^+$ factorization with factors containing $A^\star$ and $B^\star$.*

The next lemma we state a little different than it has been in [32].

Lemma 6.8. [32, Lemma 5] *Let $G = AB$ with $A, B \max^- G$. Suppose that $A \cap T$ is maximal in T and that $TN_G(A \cap T) = G$. Then $G^\star = A^\star B^\star$ is a $\max^+$ factorization.*

Corollary 6.9. *Suppose that G acts primitively on a set Ω and that there is a regular subgroup X in G which does not contain T. Let $x \in \Omega$ and set $A = G_x$. Let B be a $\max^-$ subgroup of G which contains X. Then $G^\star$ has a $\max^+$ factorization with factors containing $A^\star$ and $B^\star$. Moreover, $G^\star = BT$ and $B = B^\star$.*

Proof. Lemma 6.7 implies the first statement of the corollary. The second is a consequence of the fact that $G = AT$, as T is transitive. Then

$$G^\star = AT \cap BT = G \cap BT = BT$$

and $B = B^\star$, the assertion. $\square$

In the case that G as well as T act primitively on the set Ω, then we obtain better results as a corollary to Lemma 6.8 and Corollary 6.9.

Table 7
Table of exceptions

T	$A \cap T$	$B \cap T$
$\mathrm{L}_{2m}(q)$, $(q-1,m) \neq 1$	$\mathrm{stab}(V_1 \oplus V_{2m-1})$	$N_T(\mathrm{PSp}_{2m}(q))$
$\mathrm{P}\Omega^+_{2m}(q)$, q, m odd	$N_T(\mathrm{GL}_m(q)/\langle -1\rangle)$	N_1
$\mathrm{P}\Omega^+_8(3)$	$\Omega_7(3)$	$\Omega^+_6(3).2$
$\mathrm{P}\Omega^+_8(3)$	$\Omega^+_8(2)$	P_{ij}, $i, j \in \{1,3,4\}$
$\mathrm{P}\Omega^+_8(3)$	$2^6 : \mathrm{A}_8$	P_i, $i \in \{1,3,4\}$
M_{12}	A_5	M_{11}

Corollary 6.10. *Suppose that G as well as T act primitively on a set Ω and that there is a regular subgroup X in G which does not contain T. Let $x \in \Omega$, set $A = G_x$ and let B be a* $\max^-$ *subgroup of G which contains X. Then $G^\star = A^\star B^\star$ is a* $\max^+$ *factorization, where $G^\star = BT$ and $B = B^\star$.*

The main theorem of [32] is as follows:

Theorem 6.11. [32] *Let G be an almost simple group with socle T, and let $G = AB$ be a* $\max^-$ *factorization of G. Then $(G^\star, A^\star, B^\star)$ as above is a* $\max^+$ *factorization unless $(T, A \cap T, B \cap T)$ are as in the table of exceptions.*

Let (G, Ω) be a primitive permutation group which has a regular subgroup X. Let $A = G_\omega$ with $\omega \in \Omega$ and let B be a $\max^-$ subgroup of G which contains X. If T acts primitively on Ω, then B is already maximal in G by Corollary 6.10.

Now assume that T does not act primitively on Ω. Moreover, assume that $(T, A \cap T, B \cap T)$ are not as listed in the table of exceptions. Then $G^\star = A^\star B^\star$ is a maximal factorization by Theorem 6.11. Moreover, $B = B^\star$. Hence, in this case we may choose $G = G^\star$.

If $(T, A \cap T, B \cap T)$ are as listed in the table of exceptions, then we cannot assume that $G = AB$ is a maximal factorization.

7. The alternating and symmetric groups

In this section we prove Theorem 1.

J. Wiegold and A.G. Williamson determined the exact factorizations of the alternating and the symmetric groups, see [38]. We go through their classification to obtain the list of the primitive actions of A_n and S_n which admit a regular subgroup as given in Theorem 1. To simplify the proof we will adopt some of their notation. They suppose an exact factorization $G = KH$ for $G = \mathrm{A}_n$ or S_n (attention: our G and K are their K and G, respectively). They introduce some special prime r (they call it p): for $n \geqslant 8$, let r be the largest prime less than $n-2$. Set $r = 3$ for $n = 5$; and $r = 5$ for $n = 6$ or 7. As r divides $|G|$, it must divide $|H|$ or $|K|$. The authors suppose that r divides $|H|$. They denote the H-orbit containing the support of an r-cycle by Γ. Let $\Lambda = \Delta \setminus \Gamma$ and set $k = |\Lambda|$ (attention: our Δ differs from theirs).

Suppose $G = \mathrm{A}_n$. Suppose first that $H^\Gamma = \mathrm{A}_n^\Gamma$. Then according to [38, Theorem A] $H^\Lambda = 1$ and K is sharply k-transitive on Δ, for some $k \in \{1, \ldots, 5\}$. If $k = 1$, then H is maximal in A_n. Let Ω be the set of cosets of H in A_n and let X be the subgroup K. Then (a.a) holds. A regular subgroup K of A_n is never maximal in A_n. (Recall: $C_{\mathrm{S}_n}(K) \cong K$; therefore, if K is not abelian, then it is not maximal in A_n. If K is abelian, then use induction on $|\Delta|$.) Suppose $k \geqslant 2$. Then H is not maximal in A_n. The sharply k-transitive groups have been classified: If $k = 2$, then $n = p^e$ is a prime power and

$$K \cong \mathrm{AGL}_1\bigl(p^e\bigr),$$

see [22, XII, Theorem 9.1]. If $k = 3$, then $n = p^e + 1$ with p a prime and either

$$K \cong \mathrm{PGL}_2\bigl(p^e\bigr)$$

or p is odd and $e = 2m$ is even and

$$G \cong M\bigl(p^e + 1\bigr),$$

see [22, XI, Theorems 2.1 and 2.6]. If $k = 4$, then K is one of the groups

$$\mathrm{S}_4, \quad \mathrm{S}_5, \quad \mathrm{A}_6, \quad \mathrm{M}_{11},$$

see [22, XII, Theorem 3.3]. If $k = 5$, then K is one of the groups

$$\mathrm{S}_5, \quad \mathrm{S}_6, \quad \mathrm{A}_7, \quad \mathrm{M}_{12},$$

see [22, XII, Theorem 3.4]. By the theorem of [29] M_{11} and M_{12} are maximal subgroups of A_{11} and A_{12}, respectively, and we obtain (a.b.b) and (a.b.c) of our assertion. If $k = 2$, then $K \cong \mathrm{AGL}_1(p^e)$ is precisely a maximal subgroup of A_n, if $n = p^e = 2$. As $n \geqslant 5$, this is impossible. Let $k = 3$. If $K \cong \mathrm{PGL}_2(p^e)$, then K is a subgroup of A_n if and only if $p = 2$ and K is maximal in A_n only if $p^e = 2$ and therefore $n = 3$, which is impossible. Suppose $K \cong M(p^e + 1)$ is maximal in A_n. Then $e = 2$ and $p = 3 \equiv 3(4)$. According to [29] $M(p^2 + 1)$ is maximal, indeed. This gives (a.b.a).

Next suppose that $H^\Gamma = \mathrm{S}_n^\Gamma$. Then by [38, Theorem A] K is k-homogeneous but not k-transitive on Δ. If H is maximal in A_n, then

$$H^\Lambda = \mathrm{S}_n^\Lambda$$

and one of the cases listed in (a.d) holds. Now suppose that K is maximal in A_n. Then by [38] and [29] $K = G_\omega$ is exactly as one of the groups listed in (a.c). In items (a.c.c) and (a.c.d) we have $p \neq 7, 11, 23$ as

$$\mathrm{L}_2(7) \leqslant 2^3 : \mathrm{L}_3(2) \leqslant \mathrm{A}_8, \qquad \mathrm{L}_2(11) \leqslant \mathrm{M}_{11} \leqslant \mathrm{A}_{11}, \qquad \mathrm{L}_2(23) \leqslant \mathrm{M}_{24} \leqslant \mathrm{A}_{24}$$

and

$$\mathrm{Frob}(7:3) \leqslant \mathrm{L}_2(7) \leqslant \mathrm{A}_7, \qquad \mathrm{Frob}(11:5) \leqslant \mathrm{L}_2(11) \leqslant \mathrm{A}_{11}, \qquad \mathrm{Frob}(23:11) \leqslant \mathrm{M}_{23} \leqslant \mathrm{A}_{23}.$$

If $H^\Gamma \neq \mathrm{A}_n^\Gamma, \mathrm{S}_n^\Gamma$, then $n = 8$, $k = 3$, $K \cong 2^3 : \mathrm{L}_3(2)$ and $|H| = 15$ (see [38, Theorem A]), so K, but not H is maximal in A_n and (a.e) holds.

Now suppose $G = \mathrm{S}_n$. As S_n acts faithfully on Ω neither K nor H contains A_n.

Suppose $H^\Gamma = \mathrm{A}_n^\Gamma$. Then H is not maximal in S_n. According to [38, Theorem A] either K contains

$$K \cap \mathrm{A}_n^\Delta$$

as a sharply k-transitive subgroup of index 2, for some $k \in \{1, 2, 3\}$, or K is sharply k-transitive with $k \in \{2, \ldots, 5\}$. In the first case, we get $k = 3$. If

$$K \cap \mathrm{A}_n \cong \mathrm{PGL}_2\bigl(p^e\bigr),$$

then $p = 2$ and $e \leqslant 2$. As $n \geqslant 5$, $e = 2$ and $K \cap \mathrm{A}_n \cong \mathrm{PGL}_2(4) \cong A$, a contradiction. Hence,

$$K \cap \mathrm{A}_n \cong M\bigl(p^e + 1\bigr)$$

with p odd and e even. The fact that $|K : K \cap \mathrm{A}_n| = 2$ yields that $e = 2$ and $p \equiv 3(4)$. So,

$$G_\omega \cap \mathrm{A}_n = K \cap \mathrm{A}_n$$

is as listed in (a.b.a) and (b.a) holds. If K is sharply k-transitive with $k \in \{2, \ldots, 5\}$, then $k \neq 4, 5$. If $k = 3$, then maximality of K in S_n forces $K \cong \mathrm{PGL}_2(p)$ with p a prime congruent 3 modulo 4—notice that K is maximal in S_n for $p = 7, 11, 23$—and (b.b.a) holds. If $k = 2$, then $K \cong \mathrm{Frob}(p : p - 1)$ and this group is always maximal in S_n, see the theorem of [29]. Hence (b.b.b) holds.

Now suppose $H^\Gamma = \mathrm{S}_n^\Gamma$. Then K is sharply k-transitive, for some $k \in \{1, \ldots, 5\}$ and $H^\Lambda = 1$ or K is k-homogeneous but not k-transitive on Δ, for some $k \in \{1, \ldots, 5\}$. Suppose the first case. If $k = 1$, then H, but not K is maximal and we obtain one of the groups listed in (b.a). If $k \geqslant 2$, then H is not maximal. If K is maximal in S_n, then we obtain the same n and Ω as in (b.b), but a different regular subgroup $X = H$. But note that $X = H \cong \mathrm{S}_{n-k}$ whether $H^\Gamma = \mathrm{A}_n^\Gamma$ or $H^\Gamma = \mathrm{S}_n^\Gamma$. Now suppose that K is k-homogeneous but not k-transitive on Δ. Then K is never maximal in S_n and H is maximal if and only if $H^\Lambda = \mathrm{S}_n^\Lambda$. Hence, (b.a) holds with $G_\omega \cap \mathrm{A}_n$ as in (a.d).

If $H^\Gamma \neq \mathrm{A}_n^\Gamma, \mathrm{S}_n^\Gamma$, then else $n = 6$ or $n = 8$. If $n = 8$, then neither H nor K is maximal and if $n = 6$, then (b.c) holds. This completes the proof of the theorem in the case that $G = \mathrm{A}_n$ or S_n.

Now assume that $n = 6$ and $G \neq \mathrm{A}_6, \mathrm{S}_6$. Checking the Atlas [11, p. 4] we see that (G, Ω, X) is as in (c).

8. The sporadic groups

Next we prove Theorem 2.

Suppose that T is a sporadic group such that a maximal subgroup A of G admits a complement X in G. Let B be a max$^-$ subgroup of G containing X. Unless $G = \mathrm{Aut}(\mathrm{M}_{12})$ and $A \cap T \cong \mathrm{A}_5$ or M_{11} Theorem 6.11 gives a subgroup $T \leqslant G^\star \leqslant G$ such that $G^\star = TB$, $A^\star = A \cap G^\star$, and $G^\star = A^\star B$ is a maximal factorization. Assume the former case, i.e. $G = \mathrm{Aut}(\mathrm{M}_{12})$. As A is maximal in G, it follows that $A \cap T \cong \mathrm{A}_5$, $A \cong \mathrm{S}_5$, and $B \cong \mathrm{M}_{11}$ and $|\Omega| = 2^4 \cdot 3^2 \cdot 11$. But there is no subgroup of order $|\Omega|$ in M_{11}, see [11, p. 18]. Hence, we may always assume that B is a maximal subgroup of G which does not contain T and we are able to quote Theorem C of [31]:

According to the list of the maximal factorizations of the sporadic simple groups given in [31, Table 6], T is isomorphic to one of the following groups:

$$\mathrm{M}_{11},\quad \mathrm{M}_{12},\quad \mathrm{M}_{22},\quad \mathrm{M}_{23},\quad \mathrm{M}_{24},\quad \mathrm{J}_2,\quad \mathrm{HS},\quad \mathrm{He},\quad \mathrm{Ru},\quad \mathrm{Suz},\quad \mathrm{Fi}_{22},\quad \mathrm{Co}_1.$$

We are going to consider each case in turn.

(1) Suppose $T \cong \mathrm{M}_{11}$. Then $G = T$ and by [31, Table 6] A is isomorphic to one of the groups:

$$\mathrm{L}_2(11),\quad \mathrm{M}_{10},\quad \mathrm{M}_9.2.$$

If $A \cong \mathrm{M}_{10}$, then (a.a) of the assertion holds, see also [27, Theorem 1.1].

Now let $A \cong \mathrm{M}_9.2$. Then $|\Omega| = |G : A| = 5 \cdot 11$ and the Frobenius subgroup $\mathrm{Frob}(11:5)$ of T is clearly a complement to A in G, as stated in (a.b).

Last assume $A \cong \mathrm{L}_2(11)$. Then $|\Omega| = 12$ is divisible by 4. Therefore the subgroups A and X of M_{11} contain involutions. This contradicts Lemma 6.2 as M_{11} does contain just one conjugacy class of involutions, see [11, p. 18].

(2) Now suppose $T \cong \mathrm{M}_{12}$. By [31, Table 6] $T \cap A$ is isomorphic to one of the groups:

$$\mathrm{M}_{11},\quad \mathrm{L}_2(11),\quad \mathrm{M}_{10}.2,\quad \mathrm{M}_9.\mathrm{S}_3,\quad 2 \times \mathrm{S}_5,\quad 4^2:\mathrm{D}_{12},\quad \mathrm{A}_4 \times \mathrm{S}_3.$$

Let $T \cap A \cong \mathrm{M}_{11}$. Then $G = T$. We show that there is a complement X_1 in T. There is one conjugacy class of maximal subgroups M in M_{12} which are isomorphic to $\mathrm{A}_4 \times \mathrm{S}_3$. Let X_1 be the subgroup of M isomorphic with $2^2 \times 3$. Then every involution and every subgroup of order 3 of X_1 is contained in a maximal subgroup of T which is isomorphic to $\mathrm{L}_2(11)$, see [11, p. 33]. Therefore they act semiregularly on Ω and X_1 is regular on Ω, see also [27, Theorem 1.1].

We just checked that $O_2(M)$ acts fixed point freely on Ω. There are four subgroups of order 3 in $M/O_2(M)$, precisely one fixes the three $O_2(M)$-orbits on Ω and one centralizes $O_2(M)$. Let X_2 be the extension of $O_2(M)$ by a subgroup S of order 3 which permutes the three orbits and which does not centralize $O_2(M)$. Then X_2 is a regular subgroup isomorphic to A_4.

Notice that $M/O_2(M)$ induces the full symmetric group S_3 on the set consisting of the three $O_2(M)$-orbits $\{O_1, O_2, O_3\}$ on Ω and every Sylow 2-subgroup of M fixes one of these orbits. As the latter is an abelian group, the stabilizer S of O_1 in M induces a Klein-four group on O_1 and $O_2(M)$ acts faithfully on O_1. Let $O_2(M) = \langle i, j\rangle$ and let k be an involution in M which inverts an element s of order 3 of X_1 and which fixes O_1 and acts on O_1 as the element j. Then $X_3 = \langle i, k, s\rangle$ is a regular subgroup and is isomorphic to $2 \times \mathrm{S}_3$.

According to [11, p. 33] the square of every element of order 4 of T fixes an element in Ω. This proves that the Sylow 2-subgroups of a regular subgroup X are elementary abelian of order 4. Thus every regular subgroup is isomorphic to X_1, X_2 or X_3, which shows (b.a) of the assertion.

Let $T \cap A \cong \mathrm{L}_2(11)$. Then $T \cap A$ is maximal in T and $|\Omega| = 2^4 \cdot 3^2$. We claim that there is a regular subgroup X in T. Therefore, we may assume $G = T$. Let $H \leqslant T$ be the stabilizer in T of an element in $\mathcal{P}$, so $H \cong \mathrm{M}_{11}$. As A is maximal in T, it is transitive on $\mathcal{P}$. Let $X \cong 3^2 : Q_8.2$ be a complement in H to $A \cap H$, see (a.b). As $T = AH = AX$, indeed X is a complement to A in G. We get from [11, p. 33] that every regular subgroup of G is already isomorphic to X, which shows (b.b) of the assertion.

If $T \cap A \cong \mathrm{M}_{10}.2, \mathrm{M}_9.\mathrm{S}_3, 2 \times \mathrm{S}_5, 4^2:\mathrm{D}_{12}$ or $\mathrm{A}_4 \times \mathrm{S}_3$, respectively, then $|\Omega| = |G:A| = 6 \cdot 11$, $2^2 \cdot 5 \cdot 11$, $2^2 \cdot 3^2 \cdot 11$, $3^2 \cdot 5 \cdot 11$, $2^3 \cdot 3 \cdot 5 \cdot 11$. In all cases either $T \cap B \cong \mathrm{M}_{11}$ or $\mathrm{L}_2(11)$, see [31, Table 6]. But there is no subgroup of one of these orders in these groups, see for instance [11, pp. 7, 11].

(3) Now let $T \cong \mathrm{M}_{22}$. Then $G = \mathrm{Aut}(\mathrm{M}_{22})$ and either $A \cong \mathrm{L}_2(11):2$ or $A \cong \mathrm{M}_{21}:2 \cong \mathrm{L}_3(4):2$, see [31, Table 6]. If $A \cong \mathrm{L}_2(11):2$, then $|\Omega|$ is divisible by 4 and therefore X contains an involution of T. This contradicts Lemma 6.2 as M_{22} has just one conjugacy class of involutions, see [11, p. 40].

Hence, $A \cong \mathrm{L}_3(4):2$ and $|\Omega| = |G:A| = 2 \cdot 11$. Let Y be a subgroup of order 11 of T. Then $N_T(Y)/Y \cong \mathbb{Z}_5$ and $N_G(Y)/Y \cong \mathbb{Z}_{10}$, see for instance [11, p. 39]. Let $R = \langle r\rangle$ be a Sylow 2-subgroup of $N_G(Y)$. Since, $C_G(R)$ is divisible by 5, it follows that r is of type 2C in G in the Atlas-notation, see [11, p. 40], while involutions in $A \setminus A'$ are of type 2B, see [11, pp. 24 and 40]. This shows that $X = Y:R$ and A intersect trivially. As there are no elements of order 22 in G, see [11, p. 40], every regular subgroup is isomorphic to X and (c) holds.

(4) Now let $T \cong \mathrm{M}_{23}$. Then, as $\mathrm{Out}(T) = 1$, $T = G$ and by [31, Table 6] A is isomorphic to one of the groups:

$$\mathrm{M}_{22}, \quad \mathrm{M}_{21}:2, \quad 2^4:\mathrm{A}_7, \quad \mathrm{Frob}(23:11).$$

If $A \cong \mathrm{M}_{22}$, then clearly, (d.a) holds, see also [27, Theorem 1.1].

Now suppose $A \cong \mathrm{M}_{21}:2$ or $2^4:\mathrm{A}_7$. Then $|\Omega| = 11 \cdot 23$ and the order of A is not divisible by 11 or 23. Therefore, clearly, every Frobenius subgroup Frob(23 : 11) of T is a complement to A, which shows (d.b) and (d.d) as well as (d.c).

(5) Finally suppose $T \cong \mathrm{M}_{24}$. Then $G = T$ and by [31, Table 6] A is isomorphic to one of the groups:

$$\mathrm{M}_{23}, \quad \mathrm{L}_2(23), \quad \mathrm{M}_{22}:2, \quad \mathrm{M}_{21}:\mathrm{S}_3, \quad \mathrm{M}_{12}:2, \quad 2^6:3{}^{\cdot}\mathrm{S}_6,$$
$$2^6:\big(\mathrm{L}_3(2)\times \mathrm{S}_3\big), \quad \mathrm{L}_2(7), \quad 2^4:\mathrm{A}_8.$$

Let $A \cong \mathrm{M}_{23}$ and $\Omega = \mathcal{P}$. Let $\Delta = (\mathcal{P}_1, \mathcal{P}_2)$ be a partition of $\mathcal{P}$ into two dodecads, i.e. two 12-element sets $\mathcal{P}_1$ and $\mathcal{P}_2$ which are elements of the Golay code. Then the stabilizer D of Δ in T is isomorphic to $\mathrm{M}_{12}:2$ and $D' \cong \mathrm{M}_{12}$ acts faithfully on both sets $\mathcal{P}_1$ and $\mathcal{P}_2$, while every element in $D \setminus D'$ interchanges $\mathcal{P}_1$ and $\mathcal{P}_2$. Let Z be a subgroup of D' which acts regularly on $\mathcal{P}_1$, see (b.a). By (b.a) Z is contained in a maximal subgroup M of D' with $M \cong \mathrm{A}_4 \times \mathrm{S}_3$. Then $N_D(M) \cong \mathrm{S}_4 \times \mathrm{S}_3$, which shows if $Z \cong 2^2 \times 3$, A_4 or $2 \times \mathrm{S}_3$, then we find an involution i normalizing Z in $D \setminus D'$. Hence $X = Z:\langle i\rangle$ acts regularly on S and X is isomorphic to one of the following groups:

$$\mathrm{D}_8 \times 3, \quad \big(2^2 \times 3\big):2 \text{ or } \mathrm{S}_4.$$

According to [11, p. 96] there is an element t in T of order 12 which act fixed point freely on Ω and which, moreover, is contained in a maximal subgroup U of T isomorphic to $\mathrm{L}_2(23)$. There is only one conjugacy class of involutions in U. Therefore, $N_U(\langle t\rangle) \cong \mathrm{D}_{24}$ is regular on Ω. Now let $U \cong 2^6:(\mathrm{L}_3(2)\times \mathrm{S}_3)$ be the stabilizer in T of a trio of $\mathcal{P}$ (that is a partition of $\mathcal{P}$ into three octads Q_1, Q_2 and Q_3), let N be a complement to $O_2(U)$ in U and let a be an element of order 6 in N such that a^3 is in $N^\infty \cong \mathrm{L}_3(2)$. Then the centralizer C of a^3 in $O_2(U)$ is of order 2^4. Moreover, the stabilizer of Q_1 in U induces a group isomorphic to $2^3:\mathrm{L}_3(2)$ on Q_1 and the kernel K_1 of the action of $O_2(U)$ on Q_1 is of order 2^3. Further K_i, $1 \leqslant i \leqslant 3$, is an irreducible module for N^∞ and $K_1^s \cap K_1 = 1$ for $s = a^2$. It follows that $L_i := C_{K_i}(a^3)$ are three disjoint lines in C. There are two more lines L_4, L_5 in C such that L_i and L_j are pairwise disjoint. As s permutes the three lines L_1, L_2, L_3 it fixes this set of five lines and therefore s stabilizes L_4 and L_5. This shows that s normalizes a subgroup R of order 4 in C. Then $R:\langle a\rangle \cong 2\times \mathrm{A}_4$ acts regularly on Ω.

Let Y be an arbitrary regular subgroup. Then $|Y|=24$ and $O_2(Y)$ is a group of order 4 or 8. Assume first $|O_2(Y)|=4$. Then $|O_{2,3}(Y)|=12$. If $O_2(Y)$ is cyclic, then $O_3(Y)$ is of order 3. According to [11, p. 96] we have $N=N_T(O_3(Y))\cong \mathrm{L}_3(2)\times \mathrm{S}_3$, N stabilizes a trio (Q_1,Q_2,Q_3) of $\mathcal{P}$ and $N^\infty \cong \mathrm{L}_3(2)$ fixes Q_1, Q_2 and Q_3. Hence in this case $Y\cong(4\times 3):2$ and every Sylow 2-subgroup is dihedral, so $Y\cong \mathrm{D}_8\times 3$ or D_{24}. Now assume that $O_2(Y)$ is elementary abelian. If $O_{2,3}(Y)\cong \mathrm{A}_4$, then $Y\cong \mathrm{S}_4$. Notice that every involution in N which centralizes N^∞ fixes a point in Ω according to [11, p. 96]. Thus if $O_{3,2}(Y)\cong 2^2\times 3$, then we obtain that Y is isomorphic to $(2^2\times 3):2$. Now assume $|O_2(Y)|=8$. If $O_2(Y)\cong \mathrm{Q}_8$, then Y is a subgroup of N, which is not possible. If $O_2(Y)$ is abelian of exponent 4, then every element of order 3 centralize $O_2(Y)$, which is not possible. As $O_2(Y)$ cannot be cyclic, we may assume that $O_2(Y)$ is elementary abelian. Then $Y\cong 2\times \mathrm{A}_4$, one of the listed groups. This proves (e.a).

Now let $A\cong \mathrm{L}_2(23)$. Let H be the stabilizer in T of a point in $\mathcal{P}$. Then $A\cap H\cong \mathrm{Frob}(23:11)$ and there are complements X_1, X_2 in H to $A\cap H$ with $X_1\cong 2^4:\mathrm{A}_7$ and $X_2\cong \mathrm{M}_{21}:2$ by (d.d); clearly, they are also complements to A in T.

Now let $Y\leqslant T$ be an arbitrary regular subgroup and C a maximal subgroup of T containing Y. Then C is isomorphic to one of the following groups

$$\mathrm{M}_{23},\quad \mathrm{M}_{22}:2,\quad 2^4:\mathrm{A}_8,\quad \mathrm{M}_{21}:\mathrm{S}_3$$

and $A\cap C$ is isomorphic to

$$23:11,\quad \mathrm{D}_{22},\quad \text{group of order 8},\quad \mathbb{Z}_3,$$

respectively, see [31, Table 6]. Application of (c) and (d) of this theorem, then implies $Y\cong X_1$ or X_2. This proves (e.b).

Now assume that A is isomorphic to one of the remaining groups, that is

$$A\cong \mathrm{M}_{22}:2, \mathrm{M}_{21}:\mathrm{S}_3, \mathrm{M}_{12}:2, 2^6:3^{\cdot}\mathrm{S}_6, 2^6:\bigl(\mathrm{L}_3(2)\times \mathrm{S}_3\bigr), \mathrm{L}_2(7), 2^4:\mathrm{A}_8.$$

In all these cases $|\Omega|$ is divisible by 23, but not a divisor of $11\cdot 23$:

$$|\Omega| = 2^2\cdot 3\cdot 23, 2^3\cdot 11\cdot 23, 2^3\cdot 7\cdot 23, 7\cdot 11\cdot 23, 3\cdot 5\cdot 7\cdot 11\cdot 23, 2^7\cdot 3\cdot 5\cdot 11\cdot 23, 3\cdot 11\cdot 23,$$

respectively. Moreover, in all these cases $B\cong \mathrm{M}_{23}$ or $\mathrm{L}_2(23)$, see [31, Table 6]. As the only proper subgroup of M_{23} as well as of $\mathrm{L}_2(23)$ divisible by 23 is of order $11\cdot 23$, see for instance [11, pp. 71 and 15], we obtain a contradiction.

(6) Suppose $T\cong \mathrm{J}_2$. By [31, Table 6] $T\cap A$ is isomorphic to one of the groups:

$$\mathrm{U}_3(3),\quad \mathrm{A}_5\times \mathrm{D}_{10}\quad \text{or}\quad 5^2:\mathrm{D}_{12}.$$

First let $T\cap A\cong \mathrm{U}_3(3)$ and let $G=\mathrm{Aut}(T)$. Then $|\Omega|=100=2^2\cdot 5^2$, and every maximal subgroup $B\cong 5^2:(4\times \mathrm{S}_3)$ of G acts transitively on Ω, see [31, Table 6]. Let $X_1=O_{5,2}(B)$. We claim that X acts regularly on Ω. According to [31, (6.6)] $T\cap A\cap B\cong \mathrm{S}_3$. Let i be an involution in X_1. Then i is the central element in $X_1/O_5(X_1)$. This shows that X_1 and $T\cap A$ intersect trivially and that $|X_1\cap A|\leqslant 2$. Hence X_1 and A intersect trivially and X_1 is a complement to A in G. Notice that $X_1/O_5(X_1)$ acts as the central element of $\mathrm{Aut}(O_5(X_1))\cong \mathrm{GL}_2(5)$ on $O_5(X_1)$.

Let X_2 be the extension of $O_5(X_1)$ by a non-central element of order 4 of B. Then X_2 is regular as well.

Assume there is a regular subgroup Y which is not isomorphic to X_1 or X_2. Then $|O_5(Y)| = 5^2$ and therefore, we may assume $Y \leqslant B$. As $Y \ncong X_i$, $i = 1, 2$, it follows that $Y/O_5(Y) \cong 2^2$ contains an involution i with $i^G \cap A \neq \emptyset$ in contradiction with $Y \cap A = 1$. This proves (f).

If $T \cap A \cong \mathrm{A}_5 \times \mathrm{D}_{10}$, then $|\Omega| = 2^4 \cdot 3^2 \cdot 7$ and therefore $|T \cap X|$ is divisible by 2. But this yields a contradiction to Lemma 6.2, as $T \cap A$ contains both classes of involutions of J_2 (see the permutation character of $T \cap A$ given in [11, p. 42]).

Now assume $T \cap A \cong 5^2 : \mathrm{D}_{12}$. Then $G = \mathrm{Aut}(T)$, $|\Omega| = 2^5 \cdot 3^2 \cdot 7$ and $B \cong \mathrm{U}_3(3) : 2$, see [31, Table 6]. Again $|T \cap X|$ is of even order, which contradicts Lemma 6.2, as the only class of involutions in $T \cap B$ is contained in $T \cap A$, see [32, (6.6)].

(7) Let $T \cong \mathrm{HS}$. Then $T \cap A$ is isomorphic to one of the groups (see [31, Table 6]):

$$\mathrm{M}_{22}, \quad \mathrm{U}_3(5):2, \quad 5:4 \times \mathrm{A}_5, \quad 5^{1+2}:8:2.$$

Suppose first $T \cap A \cong \mathrm{M}_{22}$ and let $G = \mathrm{Aut}(T)$. Then Ω is the set of points of the Higman–Sims graph, so $|\Omega| = 100 = 2^2 \cdot 5^2$ and, moreover, every maximal subgroup $B \cong 5:4 \times \mathrm{S}_5$ of G acts transitively on Ω, see [31, Table 6]. Let X_1 be a subgroup of B such that $X_1 \cap T \cong 5 \times 5 : 2$ and $X_1 \cong 5 \times 5 : 4$. Then $X_1 \cap B^\infty \cong 5 : 2$. We claim that X_1 and A intersect trivially. According to [31, (6.7)], the elements of order 5 in A are of type 5C while those in B are of type 5A or 5B (we use Atlas-notation again). Hence, A and X_1 intersect in some 2-group. Assume that there is an involution in $A \cap X_1$. Then, by the choice of X_1, the involution is already contained in T and therefore of type 2B, see [11, p. 80]. But we can read from the permutation character of $T \cap A$ that there is exactly one class of involutions in $T \cap A$ and they are of type 2A, see [11, pp. 80, 81]. This contradiction yields that A and X_1 intersect trivially.

Consider $N_B(O_5(X_1)) \cong 5 : 4 \times 5 : 4$ and call this group M. It contains three conjugacy classes of involutions. Let N be a subgroup of M which contains X_1 as a subgroup of index 2. Then N acts transitively on Ω and every element of Ω is fixed by an involution of N. According to [11, p. 35] each of these involutions fixes exactly 20 elements in Ω. Hence they form an orbit of length 5 in N. This shows that every diagonal involution of M acts fixed point freely on Ω, as well. Let s be an element of order 4 such that s^2 is the diagonal involution. Then the extension of $O_5(X_1)$ by $\langle s \rangle$ is a regular subgroup, as well. In this way we obtain two further non-isomorphic subgroups X_2, X_3. In these groups $O_5(X_i)$ is self centralizing and X_i fixes every proper subgroup of $O_5(X_i)$ or has orbits of length 1, 1, 4, for $i = 2, 3$.

Now let Y be an arbitrary regular subgroup of G. Then $|O_5(Y)| = 5^2$ and $Y \leqslant N_G(O_5(Y)) \leqslant C$ with $C \cong \mathrm{U}_3(5) : 2$ or $C \cong B$, see [11, p. 80]. In the first case $T \cap A \cap C \cong \mathrm{A}_7$. As C' has precisely one conjugacy class of involutions, see [11, p. 34], it follows that there is no complement to $A \cap C$ in C. Thus $C \cong B$ and $Y/O_5(Y)$ is cyclic of order 4, see [11, p. 80]. Thus Y is isomorphic to one of the groups X_1, X_2, X_3. This shows item (g) of the assertion.

Now assume $T \cap A \cong \mathrm{U}_3(5) : 2$. Then $G = T$, $|\Omega| = 2^4 \cdot 11$, $B \cong \mathrm{M}_{22}$ and $A \cap B \cong \mathrm{A}_7$. Hence, A contains involutions of the same class as the involutions in B, which contradicts, as $|\Omega|$ is even, with Lemma 6.2.

If $T \cap A \cong 5 : 4 \times \mathrm{A}_5$, then elements of both classes of involutions of HS are contained in A, see for instance [31, (6.7)]. Thus the fact that $|\Omega| = 2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ does not fit with Lemma 6.2.

Finally assume $T \cap A \cong 5^{1+2} : 8 : 2$. Then $G = \mathrm{Aut}(T)$, $B \cong \mathrm{Aut}(\mathrm{M}_{22})$ and $|\Omega| = 2^6 \cdot 3^2 \cdot 7 \cdot 11$, see [31, Table 6]. In contradiction to our assumption there is no subgroup of order $|X| = |\Omega|$ in B, see [11, p. 39]. Thus $T \not\cong \mathrm{HS}$.

(8) Now assume $T \cong \mathrm{He}$. Then $T \cap A$ is isomorphic to one of the groups (see [31, Table 6]):

$$\mathrm{Sp}_4(4) : 2, \quad 7^2 : \mathrm{SL}_2(7)$$

or $G = \mathrm{Aut}(T)$ and $A \cong 7^{1+2} : (\mathrm{S}_3 \times 6)$. The group T has precisely two classes of involutions and elements of both classes are included in the subgroup of T isomorphic to $\mathrm{Sp}_4(4) : 2$, see [11, pp. 104, 105].

Assume $T \cap A \cong \mathrm{Sp}_4(4) : 2$ and let $G = \mathrm{Aut}(T) \cong \mathrm{He} : 2$. Then $|\Omega| = 2 \cdot 3 \cdot 7^3$ and A does not contain elements of type 3B or outer involutions (of type 2C). Notice that $|A|$ is not divisible by 7, that there is an element of type 3B and 2C, say i, in a maximal subgroup B of G isomorphic to $7^{1+2} : (\mathrm{S}_3 \times 6)$ and that i inverts $O_7(B)/Z(O_7(B))$. Let $X \cong 7^{1+2} : 6$ be a subgroup of B whose elements of order 3 and 2 are of type 3B and 2C. Then X is a complement to A in G.

If Y is another regular subgroup, then, as Y contains only outer involutions it follows that $Y \cong 7^{1+2} : 6$.

Assume $T \cap A \cong 7^2 : \mathrm{SL}_2(7)$. Then $T \cap B \cong \mathrm{Sp}_4(4) : 2$, see [31, Table 6], and $|\Omega| = 2^6 \cdot 3^2 \cdot 5^2 \cdot 17$. But there is no subgroup of order $|X| = |\Omega|$ in $N_{\mathrm{Aut}(T)}(T \cap B) \cong \mathrm{Sp}_4(4) : 4$, see [11, p. 44]. Thus $T \cap A \not\cong 7^2 : \mathrm{SL}_2(7)$.

Assume $G = \mathrm{Aut}(T)$ and $A \cong 7^{1+2} : (\mathrm{S}_3 \times 6)$. Then $B \cong \mathrm{Sp}_4(4) : 4$, see [31, Table 6], and $|\Omega| = 2^9 \cdot 3 \cdot 5^2 \cdot 17$. Again there is no subgroup of order $|X| = |\Omega|$ in $N_{\mathrm{Aut}(T)}(T \cap B) \cong \mathrm{Sp}_4(4) : 4$, see [11, p. 44], in contradiction to our assumption. So (g) holds.

(9) Assume $T \cong \mathrm{Ru}$. Then $T \cap A$ is isomorphic to one of the groups (see [31, Table 6]):

$$C \cong \mathrm{L}_2(29) \quad \text{or} \quad D \cong {}^2\mathrm{F}_4(2).$$

Notice that C and D intersect in a subgroup of order 3. Clearly, both the simple groups $\mathrm{L}_2(29)$ and ${}^2\mathrm{F}_4(2)$ do not contain a subgroup of index 3. Hence, in both cases there is no regular subgroup and T is not isomorphic to Ru.

(10) Assume $T \cong \mathrm{Suz}$. Then $T \cap A$ is isomorphic to one of the groups (see [31, Table 6]):

$$\mathrm{G}_2(4), \quad \mathrm{U}_5(2), \quad 3^5 : \mathrm{M}_{11}$$

and we may assume $G = \mathrm{Aut}(T) \cong T : 2$.

Assume $T \cap A \cong \mathrm{G}_2(4)$. Then $T \cap B \cong \mathrm{U}_5(2)$ or $3^5 : \mathrm{M}_{11}$, see [31, Table 6], $|\Omega| = 2 \cdot 3^4 \cdot 11$, and B does not contain a subgroup of this order, see [11, pp. 72, 18].

Now assume $T \cap A \cong \mathrm{U}_5(2)$. Then $T \cap B \cong \mathrm{G}_2(4)$, see [31, Table 6], and $|\Omega| = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$. Notice, that there is up to conjugation only one subgroup D in $\mathrm{G}_2(4)$ whose order is divisible by $7 \cdot 13$; it is $D \cong \mathrm{L}_2(13)$, [11, p. 97]. This shows that there is no subgroup of order $|\Omega|$ in $B \cong \mathrm{G}_2(4) : 2$.

Thus $T \cap A \cong 3^5 : \mathrm{M}_{11}$ and $|\Omega| = 2^9 \cdot 5 \cdot 7 \cdot 13$. We obtain another contradiction, since again $T \cap B \cong \mathrm{G}_2(4)$, see [31, Table 6], but, as discussed in the last paragraph, there is no subgroup of order $|\Omega|$ in $B \cong \mathrm{G}_2(4) : 2$. So T is not isomorphic to Suz.

(11) Next assume $T \cong \mathrm{Fi}_{22}$. Then $T \cap A$ is either isomorphic to the Tits group ${}^2\mathrm{F}_4(2)'$ or to the unitary group $2.\mathrm{U}_6(2)$ (see [31, Table 6]).

There are three classes of involutions in T and the subgroup U of T isomorphic to $2.\mathrm{U}_6(2)$ contains elements of all three classes, see [11, p. 163].

Assume $A \cap T = U$. Then $|\Omega| = 2 \cdot 3^3 \cdot 5 \cdot 13$ and by Lemma 6.2 $G = \mathrm{Aut}(T)$, $|X \cap T| = 3^3 \cdot 5 \cdot 13$ and X is contained in a subgroup B of T isomorphic to ${}^2\mathrm{F}_4(2)'$, which is impossible, see [11, p. 74].

Hence $A \cap T \cong {}^2\mathrm{F}_4(2)'$. Then $|\Omega| = 2^6 \cdot 3^6 \cdot 7 \cdot 11$ and $X \cap T$ is isomorphic to a subgroup of U. But there is no proper subgroup of U which is divisible by $3^6 \cdot 7 \cdot 11$ and $|U| \neq |\Omega|$, see [11, p. 115]. Thus, T is not isomorphic to Fi_{22}.

(12) Finally assume $T \cong \mathrm{Co}_1$. Then $G = T$, as T does not admit outer automorphisms, and A is isomorphic to one of the groups (see [31, Table 6]):

$$\mathrm{Co}_2, \quad \mathrm{Co}_3, \quad 3.\mathrm{Suz}.2, \quad \big(\mathrm{A}_4 \times \mathrm{G}_2(4)\big).2.$$

Assume $A \cong \mathrm{Co}_2$. Then $|\Omega| = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$, and $B \cong 3.\mathrm{Suz}.2$ or $B \cong (\mathrm{A}_4 \times \mathrm{G}_2(4)).2$, see [31, Table 6]. Notice, that the only maximal subgroup of Suz which is divisible by $5 \cdot 7 \cdot 13$ is isomorphic to $\mathrm{G}_2(4)$, see [11, p. 131], and $\mathrm{G}_2(4)$ itself does not have a proper subgroup which is divisible by $5 \cdot 7 \cdot 13$, see [11, p. 97]. Thus there is no subgroup X of order $|\Omega|$.

Now assume $A \cong \mathrm{Co}_3$. Then $|\Omega| = 2^{11} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ and again $B \cong 3.\mathrm{Suz}.2$ or $B \cong (\mathrm{A}_4 \times \mathrm{G}_2(4)).2$ of T. As above, both cases are impossible.

Hence $A \cong 3.\mathrm{Suz}.2$ or $(\mathrm{A}_4 \times \mathrm{G}_2(4)).2$. Then either $B \cong \mathrm{Co}_2$ or $B \cong \mathrm{Co}_3$, see [31, Table 6]. Clearly we obtain a factorization $B = (A \cap B)X$ of B and if M and N are maximal subgroups of B containing $A \cap B$ and X, respectively, then $B = MN$ is a maximal factorization of B. This contradicts the fact that there are no maximal factorizations for Co_2 and Co_3, see [31, Table 6]. This last case finishes the proof of the theorem.

9. The exceptional groups of Lie type

In this section we prove that exceptional groups of Lie type do not possess regular subgroups.

Let T, G and Ω be as in the assertion of Theorem 3 and assume that there is a subgroup X in G which acts regularly on Ω. Let A be the stabilizer of an element in Ω and let B be a maximal subgroup of G containing X. By Theorem 6.11 we may assume that B does not contain T. Hence $G = AB$ is a maximal factorization and $T \cong \mathrm{G}_2(q)$ or $\mathrm{F}_4(q)$ according to [31, Theorem B].

Assume first $T \cong \mathrm{G}_2(q)$. Then q is a power of 3 and $A \cap T$ is one of

$$\mathrm{SL}_3(q):2, \quad \mathrm{SU}_3(q):2, \quad {}^2\mathrm{G}_2(q),$$

or $q = 4$, see [31, Theorem B].

Assume that $q = 3^n$ and that $A \cap T \cong \mathrm{SL}_3(q):2$. Then

$$|\Omega| = |G:A| = q^3\big(q^3+1\big)$$

and $B \cap T \cong \mathrm{SU}_3(q):2$ or ${}^2\mathrm{G}_2(q)$. As $B \cap T$ has a maximal factorization (Lemma 6.3), it follows with [31, Theorems A and B] that $q = 3$ and $B \cap T \cong \mathrm{SU}_3(3):2$. Then $X \cap T$ is contained in a parabolic subgroup of B, which is impossible as every element of B whose order equals the Zsigmondy prime q_6 acts irreducibly on the natural B-module.

Assume that $A \cap T \cong \mathrm{SU}_3(q):2$ or ${}^2\mathrm{G}_2(q)$. Then, again by [31, Theorem B], $B \cap T \cong \mathrm{SL}_3(q):2$, and

$$|\Omega| = |G:A| = q^3\big(q^3-1\big) \quad \text{or} \quad q^3\big(q^3-1\big)(q+1),$$

respectively. Again $X \cap T$ is contained in a parabolic subgroup of B in its action on its natural module. Again, this is impossible, as every element of B of order q_3 acts irreducibly on the natural B-module.

Hence, $q = 4$ and $A \cap T$ is one of the following groups (see [31, Theorem B])

$$\mathrm{J}_2, \quad \mathrm{SU}_3(4):2, \quad \mathrm{G}_2(2).$$

If $A \cap T \cong \mathrm{J}_2$, then $B \cap T \cong \mathrm{SU}_3(4):2$ by [31, Theorem B] and we obtain a non-trivial factorization of $B \cap T$ by Lemma 6.3, which contradicts [31, Theorem A].

Assume $A \cap T \cong \mathrm{SU}_3(4):2$. Then $B \cap T \cong \mathrm{J}_2$, see [31, Theorem B], and

$$|\Omega| = 2^5 \cdot 3^2 \cdot 7.$$

Now [31, Theorem C] implies that $X \leqslant E \leqslant B$ with $E \cong \mathrm{U}_3(3):2$ or $E \cong \mathrm{U}_3(3)$, which is impossible, as X would be of index 3 or 6 in E.

So, $A \cap T \cong \mathrm{G}_2(2)$. Then $B \cap T \cong \mathrm{SU}_3(4):2$ by [31, Theorem B], which contradicts again Lemma 6.3 and [31, Theorem A].

Now assume that $T \cong \mathrm{F}_4(q)$. Then [31, Theorem B] and Lemma 6.3 imply that q is a power of 2 and that $A \cap T \cong {}^3\mathrm{D}_4(q)$ and $B \cap T \cong \mathrm{Sp}_8(q)$. Then

$$|\Omega| = |G:A| = q^{12}\left(q^8 - 1\right)\left(q^4 - 1\right)$$

and therefore, X is divisible by the Zsigmondy prime q_8. Let E be a maximal subgroup of B containing X. According to [31, Table 2.5] E is the normalizer of a field extension (of type C_3 in [2]), but not of type $\mathrm{GU}_m(q).2$, or a classical subgroup (of type C_8 in [2]). In both cases the order of E is not divisible by $|\Omega|$, see for instance [24, (4.3.10)].

10. The classical groups

Now we prove the remaining theorems. First we present all the examples of classical groups having regular subgroups which are known to us and prove Theorems 4–7. Finally, we outline a scheme of the classification of the regular subgroups for the remaining classical groups.

From now on suppose T to be a classical group. Recall the definition of the symbol "ˆ" in Section 3 and recall also that

$$\mathrm{L}_2(4) \cong \mathrm{L}_2(5) \cong \mathrm{O}_4^-(2) \cong \mathrm{A}_5,$$
$$\mathrm{L}_2(9) \cong \mathrm{Sp}_4(2)' \cong \mathrm{O}_4^-(3) \cong \mathrm{A}_6,$$
$$\mathrm{L}_4(2) \cong \mathrm{O}_6^+(2) \cong \mathrm{A}_8.$$

Thus the primitive actions of these groups have already been studied in Theorem 1. Now we assume $T \ncong \mathrm{A}_5, \mathrm{A}_6, \mathrm{A}_8$. Nevertheless in our list of examples we include some examples with $T \cong \mathrm{A}_n$, $n \in \{5, 6, 8\}$, if they are a member of a larger family of examples.

10.1. Examples

Examples over GF(2)

Example (a). Let $T \cong \mathrm{U}_4(2)$ and let (V, f) be the natural 4-dimensional unitary GF(2)-space for T. We claim that there is a subgroup X of T which acts regularly on the set Ω of maximal totally isotropic subspaces of V. We show the following stronger result:

Lemma 10.1. *A subgroup X of $G = \mathrm{Aut}(T)$ is regular, if and only if X is extraspecial of order* 27.

Proof. We have $|\Omega| = 27$. Let P be a Sylow 3-subgroup of G and let A be the stabilizer of a maximal totally isotropic subspace of V. Then $P \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$ and if $a \in A$ has order 3, then $a^G \cap P$ is a set of three generators and their inverses in the unique abelian subgroup of P of order 27, see [11, pp. 26, 27]. Therefore the maximal subgroups of P avoiding a^G are precisely the extraspecial ones. $\square$

Moreover, notice the following.

Lemma 10.2. *Let $T \cong \mathrm{U}_4(2)$, $T \leqslant G \leqslant \mathrm{Aut}(T)$ and let A be a maximal subgroup of G such that there is a complement X to A in G. Then A and X are as in Example* (a).

Proof. This can be read off the Atlas [11, p. 26]. $\square$

Example (b). (See also Theorem 1(b.a).) Let $G = \mathrm{S}_6 \cong \mathrm{Sp}_4(2)$. Let $\Omega = \{1, \ldots, 6\}$. Then $G_1 \cong \mathrm{S}_5$ and there is a complement X to $A = G_1$ in G, with X cyclic of order 6. Clearly, there is also a regular subgroup in G which is isomorphic to S_3.

Example (c). (See also Theorem 1(b.b.a).) Mind, in this example G is not a classical group. Let $G = \mathrm{S}_7$, let A be a maximal subgroup of G isomorphic to $\mathrm{Frob}(7:6)$ and let Ω be the set of cosets of A in G. Let $\Delta = \{1, \ldots, 7\}$. Then we claim that the stabilizer X of 6 and 7 in G acts regularly on Ω. As every element in A fixes at most one element in Δ, it follows that $A \cap X = 1$. Hence X is a complement to A in G.

Example (d). (See also Theorem 1(b.b.b).) Let $G = \mathrm{SO}_6^+(2) \cong \mathrm{S}_8$, let A be a maximal subgroup of G isomorphic to $\mathrm{L}_3(2):2$ and let Ω be the set of cosets of A in G. Consider the action of $G \cong \mathrm{S}_8$ on $\Delta = \{1, \ldots, 8\}$. Then A acts sharply triply transitive on Δ. Therefore, the stabilizer X_1 of 6, 7 and 8 in G is a complement to A in G. Thus $X_1 \cong \mathrm{S}_5$ acts regularly on Ω.

Note that we find also a regular subgroup in T. Let $X_2 \leqslant T$ such that $X_2' = X_1'$ and $X_2 \cong X_1$. As every involution of $A \cap T$ acts fixed point freely on Δ it follows that X_2 is a complement to A in $\mathrm{Aut}(T)$.

We might have derived this example from the previous example in the same way Example (c) can be obtained from Example (b): Let $H = \mathrm{S}_{\{1,\ldots,7\}} \leqslant G$. Then $A \cap H \cong \mathrm{Frob}(7:6)$ and $X \cap H = X$. By Example (c) X is a complement to $A \cap H$ in H. As the involutions in $A \cap T \cong \mathrm{L}_3(2)$ are long involutions, it follows that X is a complement to A in G.

The following will be a consequence of Lemma 10.22.

Lemma 10.3. *If X is a regular subgroup, then $X \cong \mathrm{S}_5$.*

Example (e). Let $G \cong \mathrm{Sp}_6(2)$ and let $(V, (\,,\,))$ be the 6-dimensional natural GF(2)-module for G equipped with a symplectic form $(\,,\,)$. The space V contains a generalized hexagon $\mathcal{H}$ with automorphism group isomorphic to $\mathrm{G}_2(2)$ and the stabilizer A of $\mathcal{H}$ in the group G is a maximal subgroup of G isomorphic to $\mathrm{G}_2(2)$. Let Ω be the set of cosets of A in G. Then $|\Omega| = |G : A| = 120$. Let $B \cong \mathrm{SO}_6^+(2)$ be the stabilizer in G of a quadratic form of $+$ type on V with associated bilinear form $(\,,\,)$. Then $A \cap B \cong \mathrm{L}_3(2) : 2$ and $A \cap B$ is a maximal subgroup of B, see [31, (5.2.3)]. Therefore, by Example (d), there is a complement $X \cong \mathrm{S}_5$ to $A \cap B$ in B.

Also as a consequence of Lemma 10.22 we will obtain

Lemma 10.4. *If X is a regular subgroup, then $X \cong \mathrm{S}_5$.*

Example (f). Let $T \cong \Omega_8^+(2)$. Then $T = AB$ with $A \cong B \cong \mathrm{Sp}_6(2)$ and $A \cap B \cong \mathrm{G}_2(2)$, see [31, (5.15)]. As there is only one conjugacy class of subgroups isomorphic to $A \cap B$ in B we may apply Example (e) to this situation and obtain that there is a subgroup $X \cong \mathrm{S}_5$ in B which acts regularly on the set of cosets of $A \cap B$ in B. So X is a complement to A in T.

Geometric Interpretation of Example (f). Let V be the natural 8-dimensional GF(2)-module for T and let Ω be the set of non-singular 1-spaces of V. Take a maximal totally singular subspace W in V and consider W as a GF(4)-vector space. Then the subgroup $\overline{X}$ of G stabilizing $W = \langle e_1, e_2, e_3, e_4\rangle$ and $\langle f_1, f_2, f_3, f_4\rangle$ as GF(4)-vector spaces acts transitively on Ω, where $\{e_i, f_i \mid 1 \leqslant i \leqslant 4\}$ is a standard basis of V. We have $\overline{X} \cong \Gamma\mathrm{L}_2(4)$ and $X \leqslant \overline{X}$ with $X \cong \mathrm{L}_2(4) : 2 \cong \mathrm{P\Gamma L}_2(4)$ acts regularly on Ω.

In Lemma 10.22 we will prove

Lemma 10.5. *If X is a regular subgroup, then $X \cong \mathrm{S}_5 \cong \mathrm{P\Gamma L}_2(4)$.*

Example (g). Let $G = T \cong \mathrm{Sp}_8(2)$ and let A be a maximal subgroup of G isomorphic to $\mathrm{SO}_8^-(2)$. We claim that there is a subgroup X in T which acts regularly on the set of cosets Ω of A in T. There is a supplement $B \cong \mathrm{SO}_8^+(2)$ in T to A with $A \cap B \cong \mathrm{Sp}_6(2) \times 2$, see [31, (3.2.4e)]. Without loss of generality we may assume that $A \cap B$ is the stabilizer of a non-singular point in the natural module for B. Then, by Example (f) there is a subgroup $X \cong \mathrm{S}_5$ in B' which is a complement to $A \cap B'$ in B'. Thus X is a complement to $A \cap B$ in B and X is a subgroup of T which acts regularly on Ω.

Lemma 10.6. *If X is subgroup of T which is regular on Ω, then $X \cong \mathrm{S}_5$.*

Proof. Clearly, $|X| = 2^3 \cdot 3 \cdot 5$. Assume that X is solvable. Then it follows that $O_5(X) \neq 1$. Notice that the subgroups of order 5 of A are of type 5A in Atlas-notation, see [11, pp. 123, 124]. Therefore the 5-elements of X are of type 5B, see [11, p. 124], and we may assume that $O_5(X)$ is contained in B (where B is as introduced above). Then it follows that $N_T(O_5(X)) = N_B(O_5(X))$, see [11, p. 124]. Thus X is a subgroup of B and therefore a complement to $A \cap B$ in B. Lemma 10.5 implies that $X^\infty \cong \mathrm{A}_5$, which is a contradiction to the fact that X is solvable. Thus X is not solvable and therefore, $X^\infty \cong \mathrm{A}_5$.

The subgroup A contains an element of every conjugacy class of involutions of T apart from those of type a_4 in the notation of [3], see [3, (7.6) and Section 8]. Hence all the involutions in X are of type a_4. Assume that $X \cong 2 \times \mathrm{A}_5$. Then we find exactly as in the last paragraph of the proof of Lemma 10.22 an involution in X which is not of type a_4. This contradiction then yields $X \cong \mathrm{S}_5$. $\square$

Example (h). Let $G \cong \mathrm{Sp}_6(2)$. Then there is precisely one conjugacy class Ω of maximal subgroups isomorphic to $\mathrm{L}_2(8):3$ in G, see [11, p. 46]. We claim that there is a subgroup X in G isomorphic to $2^4:\mathrm{A}_5$ which acts regularly on Ω. Let B be a subgroup of G isomorphic to $\mathrm{SO}_6^-(2) \cong \mathrm{U}_4(2):2$. Then B acts transitively on Ω, see [11, p. 46], and it follows that $A \cap B \cong \mathrm{D}_{18}:3$ where A is an element in Ω. Let V be a natural G-module and consider it as a natural module for B. Then let $X \cong 2^4:\mathrm{A}_5$ be the stabilizer of an isotropic point of V in $B' \cong \Omega_6^-(2)$. So we still need to show that $X \cap A \cap B = 1$. We claim that every involution i in $A \cap B$ is contained in $B \setminus B'$. To see this notice that i centralizes precisely a 1-dimensional GF(8)-subspace of V and therefore a 3-dimensional GF(2)-subspace of V. This yields $i \in B \setminus B'$, see for instance the lemma in [31, Section 2.2.2]. Hence $X \cap A \cap B$ is a 3-group. The 3-elements of A are of type 3B and 3C in G, see [11, pp. 46, 47].

We claim that the class of 3-elements of X are of type 3A. As B contains a Sylow 3-subgroup of G, we may choose an element r in B which is of type 3A in G. It remains to show that r fixes an isotropic point in the B-module V. According to [11, p. 46] r acts faithfully on a non-isotropic line U of V and centralizes $U^\perp$ (considered as a module for G). As the 4-dimensional subspace $U^\perp$ of V considered as a B-module contains an isotropic point, r fixes an isotropic point in the latter module, which proves that the class of 3-elements of X are of type 3A. Hence $X \cap A \cap B = 1$, which yields the assertion.

Notice, that we also get $O_3(A \cap B) \cong 3_-^{1+2}$ (respectively $A \cap B$) is a complement to X in B' (respectively B).

From [11, p. 46] we derive that

Lemma 10.7. *If X is subgroup of G which is regular on Ω, then $X \cong 2^4:\mathrm{A}_5$.*

Example (i). Let $T \cong \Omega_8^+(2)$ and let (V, Q) be the natural 8-dimensional GF(2)-module for T equipped with an orthogonal form of $+$ type. Let A be a maximal subgroup of T isomorphic to A_9 and let Ω be the set of cosets of A in T. Then $|\Omega| = 2^6 \cdot 3 \cdot 5$. Let B be the stabilizer of a maximal totally singular subspace of V in T. We assume that A is chosen such that $AB = T$, see [11, p. 85]. Then $B \cong 2^6:\Omega_6^+(2) \cong 2^6:\mathrm{L}_4(2)$ and $A \cap B \cong 2^3:\mathrm{L}_3(2)$ and $A \cap O_2(B) = 1$, see the Atlas [11, pp. 85, 86]. Let K be a complement to $O_2(B)$ in B and $R \cong \mathbb{Z}_{15}$ a Singer-cycle in K. We claim that $X_1 = O_2(B):R$ acts regularly on Ω. To show this we prove that $A \cap B$ and X_1 intersect trivially. As $|X_1| = |\Omega|$, the claim then follows. By construction we have

$$X_1/O_2(B) \cap \big((A \cap B)O_2(B)/O_2(B)\big) = 1.$$

Hence $X_1 \cap A \leqslant O_2(B)$ which implies $X_1 \cap A = O_2(B) \cap A = 1$, the claim. Thus $X_1 = O_2(B):R$ acts regularly on Ω.

We claim that there is also a regular subgroup $X_2 \cong 2^4:\mathrm{A}_5$ in T. According to [11, p. 85] there is a maximal subgroup C of T with $C \cong \mathrm{Sp}_6(2)$ such that $T = AC$. Then $A \cap C \cong \mathrm{L}_2(8):3$. According to Example (h) there is a complement $X_2 \cong 2^4:\mathrm{A}_5$ in C to $A \cap C$, which proves the claim.

Last not least notice that there is a maximal subgroup D of T isomorphic to $(3 \times \mathrm{U}_4(2)):2$ which is a supplement to A in T, see for instance [11, p. 85]. We may assume that A is chosen such that the elements of order 2, 3 and 5 are of type 2B, 2E, 3A, 3D, 3E and 5A and that D is the normalizer of an element of type 3B in T, see [11, pp. 85, 86]. Then elements of order 2 and 5 in D are of type 2A, 2C, 2E and 5B. Notice that the involutions of type 2E do not centralize a subgroup of order 9, see [11, p. 86]. Therefore, in D the involutions of type 2E invert $O_3(D)$. Let

X_3 be a subgroup of D isomorphic to $(3 \times 2^4 : (5:2))^{\cdot}2 \cong 3 : (2^4 : (5:4))$. Then $X_3/O_{3,2}(X_3) \cong$ Frob$(5:4)$, which implies that $X_3 \setminus X_3 \cap D'$ does not contain any involutions. Hence, $X_3 \cap A = 1$ and X_3 acts regularly on Ω, as well.

Let $G = N_{\mathrm{Aut}(T)}(\Omega) \cong T : 2$.

Lemma 10.8. *If X is a subgroup of G which is regular on Ω, then $X \leqslant T$ and X is isomorphic to one of the groups $2^6 : \mathbb{Z}_{15}$ or $(3 \times 2^4 : (5:2))^{\cdot}2 \cong 3 : (2^4 : (5:4))$ or $O_2(X)$ is elementary abelian of order 16 and $X/O_2(X) \cong \mathrm{A}_5$.*

Proof. As G is acting on Ω we may assume that G is maximal with that property, that is $G = \mathrm{O}_8^+(2)$, see [11, p. 85]. Let X be a regular subgroup of G. Then $|X| = |\Omega| = 2^6 \cdot 3 \cdot 5$. It then follows that X is not an almost simple group, see for instance [11, p. 239].

First of all notice that $O_5(X) = 1$. Assume $O_5(X) \neq 1$. Then $C_X(O_5(X))$ is divisible by $2^4 \cdot 3 \cdot 5$. On the other hand, $C_G(O_5(X))$ is of order $2^2 \cdot 3 \cdot 5^2$ (if a in A is a 5-element, then $|C_G(a)| = 2^3 \cdot 3 \cdot 5^2$), see [11, p. 86]. Thus $O_5(X) = 1$.

Now assume $O_3(X) \neq 1$. Then X is a subgroup of $N_G(O_3(X)) \cong (3 \times \mathrm{U}_4(2)) : 2$ and X is isomorphic to $3 : (2^4 : (5:4))$, see [11, p. 26], one of the groups listed in the assertion.

Thus we may assume $O_3(X) = 1$. Let N be a minimal normal subgroup of X. Assume that N is non-abelian. Then $N \cong \mathrm{A}_5$ and $C_X(N)$ is of order 2^3 or 2^4, which contradicts the fact that the centralizer of every element of order 5 of X is of order $2^2 \cdot 3 \cdot 5^2$. Hence N is abelian and therefore $N \leqslant O_2(X) = F^*(X)$. Thus elements of order 5 of X act non-trivially on $O_2(X)$. Therefore $O_2(X)$ is of order $2^4, 2^5$ or 2^6. In the last case $X \cong 2^6 : \mathbb{Z}_{15}$, so the assertion holds.

Assume $|O_2(X)| = 2^5$. Then by the theorem of Sylow $X/O_2(X)$ contains an element x of order 15 and the centralizer of x in $O_2(X)$ is of order 2. But this is not possible as $C_G(x^3) \cong 5 \times \mathrm{A}_5$, see [11, pp. 85, 87].

Thus we may assume $|O_2(X)| = 2^4$. Then $O_2(X)$ is elementary abelian. Assume that $\overline{X} = X/O_2(X)$ is solvable. We are going to show that this is not possible. As $\overline{X}$ is solvable, it is isomorphic to a solvable subgroup of order $2^2 \cdot 3 \cdot 5$ of $\mathrm{Aut}(O_2(X)) \cong \mathrm{L}_4(2)$, so $\overline{X} \cong (3 \times 5 : 2)^{\cdot}2$. Further X is a subgroup of a maximal parabolic subgroup P of G with $P \cong 2^6 : \mathrm{S}_8$, see [11, p. 85]. By our choice of A it contains involutions of type 2A and therefore, X does not contain 2A-involutions. According to [11, p. 85] $O_2(P)$ contains 28 involutions which are not of type 2A, and which form an orbit Δ under the action of P. Thus the stabilizer of a non-2A-involution in $P/O_2(P)$ is isomorphic to $\mathrm{S}_6 \times 2$. It follows that every 3-element x of X fixes 10 elements in Δ. Therefore x centralizes a subgroup of order 2^4 in $O_2(P)$ and the centralizer C of x in $O_2(X)$ is of order 2^2. This is not possible as $O_5(\overline{X})$ acts on C. Thus $\overline{X}$ is not solvable. Therefore $\overline{X} \cong \mathrm{A}_5$ and X is isomorphic to a group listed in the assertion. This proves the lemma. $\square$

Examples over GF(4)

Example (j). Let $G = \mathrm{P\Gamma L}_3(4)$ and let A be a maximal subgroup isomorphic to Frob$(7:3) \times \mathrm{S}_3$. Then it follows from [11, p. 23] that there is a subgroup $X \cong 2^4 : (3 \times \mathrm{D}_{10})^{\cdot}2 \cong 2^4 : (3 : (5:4))$ which acts regularly on the coset space Ω of A in G. Moreover, we also get

Lemma 10.9. *If X is subgroup of G which is regular on Ω, then $X \cong 2^4 : (3 \times \mathrm{D}_{10})^{\cdot}2$.*

Example (np). Note, this is a non-primitive example. Let $G = \mathrm{Aut}(\mathrm{L}_4(4))$, let V be the natural module for T and let A be the stabilizer of a decomposition $V_1 \oplus V_3$ of V into the direct sum of a 1-dimensional subspace V_1 and a 3-dimensional subspace V_3. Then $A \cong \mathrm{GL}_3(4) : (2 \times 2)$.

Further, let K be a subgroup of index 3 in A, so $K \cong \mathrm{SL}_3(4):(2\times 2)$ and $K\cap T = \mathrm{soc}(K)$. Let Ω be the set of cosets of K in G. Notice that G does not act primitively on Ω. We claim that there is a subgroup X of G which acts regularly on Ω. We have

$$|\Omega| = |G:K| = 4^3\left(4^4-1\right).$$

Let $X \leqslant G$ with $X \cong \mathrm{P\Gamma L}_2(16)$, the normalizer in G of a field extension $\mathrm{GF}(4) \subseteq \mathrm{GF}(16)$. As in [31, Proposition B of (3.1.2)], we see that X acts transitively on the set of antiflags $V_1 \oplus V_3$ of V. Hence $X \cap A$ is a subgroup of order 3 of X. (Let $\mathcal{B}_i$ be a basis of V_i, $i = 1, 3$, then $X \cap A$ is generated by a diagonal matrix with respect to the basis $\mathcal{B}_1 \cup \mathcal{B}_3$ of V.) As these subgroups are not contained in K, we get $X \cap K = 1$. Then $|X| = |\Omega|$ yields the claim.

As a consequence of Lemma 10.22 we obtain the following.

Lemma 10.10. *If X is subgroup of G which is regular on Ω, then $X \cong \mathrm{P\Gamma L}_2(16)$.*

Example (k). Let $T \cong \mathrm{L}_4(4)$, $G = T:2 \cong \mathrm{P\Gamma L}_4(4)$ and let A be a maximal subgroup of G isomorphic to $(5\times \mathrm{L}_2(16)).4$. Then $|G:A| = 2^7\cdot 3^3\cdot 7$. Let Ω be the set of cosets of A in G. Then the stabilizer B of a 1-space U in the natural module V for T acts transitively on Ω. Let X be a subgroup of B such that $O_2(B) \leqslant X$ and $X/O_2(B)$ is the normalizer of an element of order 7 in $B/O_2(B) \cong \mathrm{\Gamma L}_3(4)$. Then $X \cong 2^6:(3\times 7:3\times 3):2$ and we claim that X acts regularly on Ω. Notice, if s is an element of order 3 in A, then s does not normalize a subgroup of order 7 of G (this can be seen by the action of s on V). If i is an involution in A, then i is in T and either $i \in A'$ or $i \in A \setminus A'$. In both cases $\dim([V,i]) = 2$, while for every involution j in $O_2(B)$ we have $\dim([V,i]) = 1$. Hence application of Lemma 6.2 yields $A \cap X = 1$ and therefore the assertion.

Notice that (a.d.a) of Theorem 1 is the same type of example for $T \cong \mathrm{A}_8 \cong \mathrm{L}_4(2)$ as well as Example (s) below.

By [11, p. 23] we get

Lemma 10.11. *If X is a subgroup of G which is regular on Ω, then $X \cong 2^6:(3\times 7:3\times 3):2$.*

Example (l). Let $T \cong \mathrm{Sp}_4(4)$, $G = T:2$ and let A be a maximal subgroup of G isomorphic to $\mathrm{P\Gamma L}_2(16)$. Let Ω be the set of cosets of A in G, then $|\Omega| = 120$. There is one conjugacy class of subgroups isomorphic to A which do not contain elements of type 2B, 3B or 5CD, see [11, p. 44]. Let A be chosen in that way. Let x be an element of type 3B of T. Then $C_T(x) \cong 3\times \mathrm{A}_5$ and there is an involution $i \in G\setminus T$ such that $X := \langle C_T(x)', i\rangle \cong \mathrm{S}_5$. According to the Atlas [11, p. 44] the elements of order 2, 3 and 5 in $C_T(x)$ are of type 2B, 3B and 5CD. Thus $C_T(x)' \cap A = 1$. As A does not contain outer involutions, it follows that X is a complement to A in G.

Checking [11, p. 44] we get

Lemma 10.12. *If X is a subgroup of G which is regular on Ω, then $X \cong \mathrm{S}_5$.*

Example (m). Let $T \cong \mathrm{Sp}_6(4)$, $G = \mathrm{Aut}(T)$ and let A_T be a maximal subgroup of T isomorphic to $\mathrm{G}_2(4)$. As there is only one conjugacy class of subgroups isomorphic to $\mathrm{G}_2(4)$ in T, see [25], we have $A := N_G(A_T)$ is isomorphic to $\mathrm{Aut}(A_T)$. Let Ω be the set of conjugates of A in G. Then

$$|\Omega| = 4^3\left(4^4-1\right).$$

We claim that there is a subgroup X in G which acts regularly on Ω. By [31, Table 2], there is a supplement B to A in G with $B \cong \mathrm{Aut}(\mathrm{L}_4(4))$ and $T \cap A \cap B \cong \mathrm{SL}_3(4):2$. As $|\Omega| = |B : A \cap B|$, it follows that $T \cap A \cap B$ is of index 2 in $B \cap A$ and therefore, $A \cap B \cong \mathrm{SL}_3(4):(2 \times 2)$. As all the subgroups isomorphic to $A \cap B$ are conjugate in B, see [24], it follows from Example (np) that there is a complement X in‘B to $A \cap B$, where $X \cong \mathrm{P\Gamma L}_2(16)$, which shows the claim.

Lemma 10.22 implies

Lemma 10.13. *If X is subgroup of G which is regular on Ω, then $X \cong \mathrm{P\Gamma L}_2(16)$.*

Example (n). Let $T \cong \Omega_8^+(4)$ and let (V, Q) be the natural 8-dimensional GF(4)-module for T equipped with an orthogonal form Q of $+$ type. Let Ω be the set of non-singular points of V. Then

$$|\Omega| = 4^3\left(4^4 - 1\right),$$

see [31, (3.6.1)]. We claim that there is a subgroup X in $G = T\langle f\rangle$ with f a field automorphism of T, which acts regularly on Ω. Let A be the stabilizer of a non-singular point in G. Then $A \cong \mathrm{Aut}(\mathrm{Sp}_6(4))$. By [31, Table 4], there is a supplement B to A in G with $B \cong A$ and $T \cap A \cap B \cong \mathrm{G}_2(4)$. As $|\Omega| = |B : A \cap B|$, it follows that $T \cap A \cap B$ is of index 2 in $B \cap A$ and therefore, $A \cap B \cong \mathrm{Aut}(\mathrm{G}_2(4))$. As there is just one conjugacy class in B of subgroups isomorphic to $A \cap B$, see [24], we obtain a complement $X \cong \mathrm{P\Gamma L}_2(16)$ to $A \cap B$ in B by Example (m), which yields the claim.

Exactly as in Example (f) we see by using Example (np) that there is a regular subgroup $X \cong \mathrm{P\Gamma L}_2(16)$ in G which fixes a maximal totally singular subspace.

The following is again a consequence of Lemma 10.22.

Lemma 10.14. *If X is subgroup of G which is regular on Ω, then $X \cong \mathrm{P\Gamma L}_2(16)$.*

Examples over GF(8)

Example (o). Let $T = \mathrm{U}_3(8)$ and let $G = T:3^2$ a subgroup of index 2 in $\mathrm{Aut}(T)$. We claim that there is a subgroup X in T which acts regularly on the set Ω of isotropic 1-spaces of V with $(V, (\,,\,))$ the natural module for T. Let A be the stabilizer of an isotropic 1-space in G and let X be the normalizer in G of a subgroup of order 19 of T. Then $X \cong \mathrm{Frob}(19:9) \times 3$ and it is a supplement to A in G according to [31, Table 3]. Since $|X| = |G : A|$, it is a complement to A in G and acts regularly on Ω as claimed.

Lemma 10.15. *If X is a regular subgroup of* $\mathrm{Aut}(T)$*, then $X \cong \mathrm{Frob}(19:9) \times 3$.*

Proof. This result follows immediately from [11, p. 66]. $\square$

Example (p). Let $T = \mathrm{U}_3(8)$ and let $G = T:3^2$ a subgroup of index 2 in $\mathrm{Aut}(T)$. Every subgroup A of G isomorphic to $19:9 \times 3$ is a maximal subgroup of G. So, G acts primitively on the set Ω of cosets of A in G and by Example (o) the stabilizer of an isotropic 1-space in G acts regularly on Ω. Using the information in [11, p. 66] we get the following.

Lemma 10.16. *If X is a regular subgroup of G, then X is the stabilizer in G of an isotropic 1-space.*

Example (q). Let $G = \mathrm{Aut}(\mathrm{U}_4(8))$ and let Ω be the set of maximal totally isotropic subspaces of the natural T-module V. Let A be the stabilizer of an element of Ω in G. Then according to [31, Theorem A] $G = AB$ with $B \cong (9 * 3^{\cdot}\mathrm{U}_3(8)).3^2$ the stabilizer of a non-isotropic 1-space U of V and $A \cap B$ is a subgroup of index 9 in the stabilizer B_Y in B of some maximal totally isotropic subspace Y of $U^{\perp}$. This yields that $A \cap B \cong 2^{3+6} : \mathbb{Z}_{63} : \mathbb{Z}_3$. Let X be the normalizer of a subgroup of order 19 in B. Then $X \cong 9 \cdot (\mathrm{Frob}(19:9) \times 3)$. Set $Y = X \cap Z(B \cap T)$, so $Y \cong \mathbb{Z}_9$. As $Y \trianglelefteq B$, B acts on the set of fixed points of Y on Ω. This shows that Y acts semiregularly on Ω and that $Y \cap A = 1$. As also $X/Y \cap AY/Y = 1$ by Example (o), it follows that X is a complement to A in G and therefore regular on Ω.

Lemma 10.17. *If X is a regular subgroup of* $\mathrm{Aut}(T)$*, then X is the normalizer of a subgroup of order* 19 *in B with B as above.*

Proof. As $AX = \mathrm{Aut}(T)$, Theorem A of [31] implies that X is a subgroup of B with B as above. As we also get a factorization of the almost simple group $\overline{B} = B/Z(B^{\infty})$, whose socle is isomorphic to $\mathrm{U}_3(8)$, [31, Theorem A] yields that $\overline{X}$ is the normalizer of a subgroup of order 19 in $\overline{B}$. Now the assertion follows, since $|X| = 9 \cdot |\overline{X}|$. □

Examples over GF(3)

Example (r). Let $T = \mathrm{U}_4(3)$ and let $T \leqslant G \leqslant \mathrm{Aut}(T)$ with $G \cong T : 2$ such that $|\mathrm{PGU}_4(3) : G| = 2$. Let A be a maximal subgroup of G such that $A \cap T \cong \mathrm{L}_3(4)$. Then $A \cong \mathrm{P\Sigma L}_3(4) = (A \cap T) : f$ with f a field automorphism of $A \cap T$. Let Ω be the set of cosets of A in G. According to [31, Theorem A] the stabilizer B in G of a totally singular line in the natural T-module is a supplement to A in G, where $B \cong 3^4 : (2 \times \mathrm{A}_6)$. Moreover, $A \cap B \cong \mathrm{A}_6$, see [31, Lemma, p. 113]. Hence, $X = O_{3,2}(B)$ is a complement to $A \cap B$ in B and therefore X is regular on Ω.

Lemma 10.18. *If X is a regular subgroup of* $\mathrm{Aut}(T)$*, then* $|O_3(X)| = 3^4$ *and* $O_2(X) = 1$.

Proof. We have $|X| = |\Omega| = 3^4 \cdot 2$. This implies $|O_3(X)| = 3^4$. Notice that T has just one class of involutions. Therefore, X only contains outer involutions of $\mathrm{Aut}(T)$. According to [11, p. 54], those outer involutions, which act on Ω do not centralize a subgroup of order 3^4. Thus $O_2(X) = 1$ and the assertion holds. □

Example (s). Let $T = \mathrm{L}_4(3)$ and $G = \mathrm{PGL}_4(3)$. Let A be a maximal subgroup of G isomorphic to $\mathbb{Z}_4 \cdot \mathrm{P\Gamma L}_2(9) \cong (\mathbb{Z}_4 \times \mathrm{PGL}_2(9)) : 2$. Let Ω be the set of cosets of A in G. Then $|\Omega| = |G : A| = 2 \cdot 3^4 \cdot 13$. Let $\widehat{A}$ be the preimage of A in $\mathrm{GL}_4(3)$ and let V be the natural module for G.

If s_1 is an element of order 3 of $\widehat{A}$, then $C_V(s_1)$ and $C_{V/\langle v\rangle}(s_1)$, with $v \in C_V(s_1)$, are 2-dimensional, and if i is an involution in A and $\hat{i}$ a preimage of i in $\widehat{A}$, then $\hat{i}$ is central or $\dim C_V(\hat{i}) = 0, 2$.

Let X be a subgroup of the stabilizer of a 1-space, say $\langle v\rangle$, of V in G isomorphic to $3^3 : ((13:3) \times 2)$. We claim that X acts regularly in Ω. Let s_2 be an element in $O_3(X)^{\#}$. Then $\dim C_{V/\langle v\rangle}(s_2) = 3$. Now assume that s_3 is an element of order 3 of X which normalizes a subgroup of order 13. Then $\dim C_{V/\langle v\rangle}(s_3) = 1$. This yields that for every element s_4 in a Sylow 3-subgroup of X outside $O_3(X)$ we have $\dim C_{V/\langle v\rangle}(s_4) = 1$. Now let i_2 be an involution in X. Then i_2 inverts $V/\langle v\rangle$. This shows that $A \cap X$ does not contain an element of order 2 or 3 and therefore $A \cap X = 1$, the claim.

Lemma 10.19. *If X is a regular subgroup of* $\mathrm{Aut}(T)$*, then* $X \cong 3^3 : ((13:3) \times 2)$.

Proof. According to [31, Theorem A] X fixes a 1-space in V and is therefore a subgroup of the parabolic group $B \cong 3^3 : \mathrm{GL}_3(3)$. Every subgroup of B of size $|\Omega| = 2 \cdot 3^4 \cdot 13$ is isomorphic to X. $\square$

Example (t). Let $T = \mathrm{PSp}_6(3)$, $G = \mathrm{Aut}(T)$ and let A be the normalizer of a field extension $\mathrm{GF}(3) \subset \mathrm{GF}(27)$ in G, so $A \cong \mathrm{L}_2(27) : 6$. Let Ω be the set of cosets of A in G. Then $|\Omega| = |G : A| = 2^7 \cdot 3^5 \cdot 5$. Let B be the stabilizer in T of a 1-space U of the natural T-module V. Then $B \cong 3^{1+4} : 2 \cdot \mathrm{Aut}(\mathrm{PSp}_4(3))$, and, moreover, $T = AB$, see [31, Table 1]. Let K be a complement to $O_3(B)$ in B, L a subgroup of K isomorphic to $2^{1+4} : (5:4)$ and set $X = O_3(B) : L$. We claim that X acts regularly on Ω. We have $|X| = |\Omega|$ and $|A \cap B|$ is of order $3^4 \cdot 2$. The assertion follows if we are able to show that elements of order 2 and 3 of A are not conjugate to elements in X. First let a be an element in $A' \cong \mathrm{L}_2(27)$ of order 3. Then a fixes a complement to U in $U^\perp$ and is therefore not contained in $O_3(B)$. Thus $X \cap O_3(B)$ is of order 1 or 3. If it were 3, then the Sylow 3-subgroups of A would be abelian, which is false. Now let a be an involution in A, then a does not fix a 1-space in V. This implies $A \cap T \cap X = 1$. As there is no involution in $X \setminus T \cap X$, it follows that $A \cap X = 1$. Notice that X fixes a complement W to U in $U^\perp$ (the eigenspace of $Z(K)$ in $U^\perp$ with eigenvalue -1), therefore, and as $[U, X] = 1$, $[U^\perp, X] \leqslant W$. Let a be an element in A of order 3. If a fixes U, then $U \leqslant [U^\perp, a]$. If a is an involution in A, then a does not fix a 1-space in V. Thus $A \cap X = 1$ and X acts regularly on Ω.

Lemma 10.20. *If X is a regular subgroup of* $\mathrm{Aut}(T)$*, then X is constructed as above, in particular* $X \cong 3^{1+4} : 2^{1+4}(5:4)$.

Proof. According to [31, Theorem A] X fixes a 1-space in V and is therefore a subgroup of the parabolic group B introduced above. Notice that all the complements to $O_3(B)$ in B are conjugate in B. Thus, as we saw above, every complement contains an element of order 3 which is conjugate to an element in A. Hence the subgroup structure of $\mathrm{Aut}(\mathrm{PSp}_4(3))$ implies the assertion. $\square$

More examples for T a linear group

Example (u). Let $G \cong \mathrm{PGL}_n(q)$ and let V be the natural module for G. Then there is a subgroup X in G, the so-called Singer-cycle, which acts regularly on the set Ω of 1-dimensional subspaces of V, see also [27, Theorem 1.1(2)(ii)]. In some cases, the normalizer of $X \cap T$ in T acts regularly, as well—for instance if $q - 1 \equiv 2(4)$ and $n = 2$. If $(n, q) = (2, 11); (2, 59); (2, 7); (2, 23)$, then there also is a regular subgroup X in $\mathrm{soc}(G) = T$ which is not a Singer-cycle: it is $X \cong \mathrm{A}_4$; A_5; D_8; S_4 or D_{24}, respectively.

Example (v). Let $T \cong \mathrm{L}_n(q)$. Suppose that the normalizer A of a Singer-cycle of T is maximal in T and that $|A| = (q^n - 1)/(q - 1)$. Then in some cases, for instance if $((q^n - 1)/(q - 1), n)$ does not divide $|G : A|$, the stabilizer of a point acts regularly on the set Ω of cosets of A in T. This happens for instance if $n = 2$ and $q - 1 \equiv 2(4)$. Then $A \cong \mathrm{D}_{q+1}$ is maximal in T if $q \neq 7$ and every subgroup of T isomorphic to $\mathrm{Frob}(q : (q - 1)/2)$ is regular on Ω. If $q = 7$, then let $G \cong \mathrm{PGL}_2(7)$ and A a Sylow 2-subgroup of G. Then again every subgroup of T isomorphic to $\mathrm{Frob}(7:3)$ is regular on Ω.

Sometimes if $|A| > (q^n - 1)/(q - 1)$ there exists nevertheless a subgroup X of the stabilizer of a point which is a complement to A in T, for instance if $(n, q) = (3, 3)$. Then $A \cong \mathrm{Frob}(13:3)$ and $X \cong 3^2 : 2^{\cdot}\mathrm{D}_8$ is a complement to A in T, so X is regular on Ω. Example (j) is also related to this type of examples, but see Example (y) as well.

Example (w). Let $G = T \cong \mathrm{L}_2(q)$ with $q = 11, 29, 59$. Then there is a maximal subgroup A in G with $A \cong \mathrm{A}_5$. Let Ω be the set of cosets of A in G. Then there is a subgroup X in G with $X \cong \mathbb{Z}_{11}, \mathrm{Frob}(29:7), \mathrm{Frob}(59:29)$, respectively, which acts regularly on Ω.

Example (x). Let $G = \mathrm{PGL}_2(11)$ or $\mathrm{L}_2(23)$. Then there is a maximal subgroup A in G which is isomorphic to S_4, see for instance [11, pp. 7, 15]. Let Ω be the set of cosets of A in G. Then every parabolic subgroup X of $T = \mathrm{soc}(G)$ with $X \cong \mathrm{Frob}(11:5)$ or $\mathrm{Frob}(23:11)$, respectively, acts regularly on Ω.

Example (y). Let $G = T \cong \mathrm{L}_5(2)$. Here we actually give two examples. In T there are maximal subgroups A and B with $A \cong \mathrm{Frob}(31:5)$ and $B \cong 2^{\cdot}(\mathrm{S}_3 \times \mathrm{L}_3(2))$. As $|G| = |A||B|$ and as $(|A|, |B|) = 1$, we have an exact factorization $G = AB$.

10.2. $O_8^+(q)$ on non-singular points

Now we start to study the orthogonal groups of dimension 8. We begin with the most difficult case, i.e. Ω is an orbit of T in its action on the set of non-singular 1-spaces in the natural module for T. In the next lemma we study a special situation of this case.

Lemma 10.21. *Let $T \cong \Omega_8^+(q)$, let V be the natural orthogonal module for T and let Ω be an orbit of T in its action on the set of non-singular 1-spaces of V. Let $T \leqslant G \leqslant \mathrm{Aut}(T)$ be such that G acts on Ω. If there is a subgroup X of G which stabilizes a maximal totally singular subspace of V and which acts regularly on Ω, then $q \in \{2, 4\}$.*

Proof. If $q = 2$ or 4, then there is a subgroup X which stabilizes a maximal totally singular subspace of V and which acts regularly on Ω, see Examples (f) and (n).

It remains to show that there is no regular subgroup in the stabilizer B of a maximal totally singular subspace, say W, for $q \neq 2, 4$. Let $q \neq 2, 4$ and assume that there is a subgroup X in B which acts regularly on Ω. As G acts on Ω, we may assume that G is the intersection of $N_{\mathrm{Aut}(T)}(T \cap B)T$ and of $N_{\mathrm{Aut}(T)}(T \cap A)T$, where A is the stabilizer of a non-isotropic 1-space of V in G. Then the p-part $|G : T|_p$ divides e or $2e$, if q is odd or even, where $q = p^e$, see Table 5 of the Atlas [11].

Note further that, if q is odd, then T has two orbits on the set of non-singular points of V and these two orbits are interchanged under the action of $\mathrm{Aut}(T) \cap \mathrm{PGL}(V)$, and if q is even, then T has just one orbit on the set of non-singular points of V, see [26, Table 3.5E] and the lemma of Witt [5, p. 81].

Notice, $T \cap A \cong \mathrm{P}\Omega_7(q)$, $G = AX$ and

$$|X| = tq^3\left(q^4 - 1\right)$$

with $t = 1/(2, q-1)$. Moreover, $A \cap B = A_Y$ is the stabilizer of v and the 3-dimensional totally isotropic subspace $Y = v^\perp \cap W$ of W in G, see [31, (3.6.1)(a)].

Write $\overline{B} = B/O_p(B)$ and use the bar convention for images of subgroups of B in $\overline{B}$. Then, clearly, $\overline{B} = (\overline{A \cap B})\overline{X}$. Moreover, X has to act transitively on the set of tuples (Y, v) with Y a 3-dimensional subspace of W and v a non-singular vector in $Y^{\perp}$. We are going to show that this never happens.

Let K be the commutator subgroup of a complement to $O_p(B)$ in B. Then

$$K \cong \mathrm{SL}_4(q)/\langle \pm 1 \rangle,$$

see [26, (4.1.20)]. By [31, Theorem A] $\overline{X} \cap \overline{K}$ is isomorphic to a subgroup of $(\mathrm{L}_2(q^2) \times \mathbb{Z}_{(q+1)}).2$. Therefore and as $q \neq 2, 4$, we get that $X \cap O_p(B) \neq 1$. Recall that $O_p(B)$ is the natural orthogonal module for $K \cong \Omega_6^+(q)$. As an element x of order q_4 of X centralizes $X \cap O_p(B)$, it follows that $X \cap O_p(B)$ is contained in a 2-dimensional subspace C of $O_p(B)$ of $-$ type, so $\overline{X}$ is isomorphic to a subgroup of the stabilizer $\overline{S}$ in $\overline{B}$ of a 2-dimensional subspace of $O_p(B)$ of $-$ type.

As $tq(q^4 - 1)$ divides the order of $\overline{X}$, in fact $\overline{X}' = \overline{S}^{\infty} \cong \mathrm{L}_2(q^2)$ or $q = 3$ and $\overline{X}' \cong \mathrm{A}_5$, see [21], and the centralizer C of $\overline{X}'$ in $O_p(B)$ is of order q^2. Then, if $q \neq 3$, $O_p(B)/C$ is the natural module for $\overline{X}' \cong \Omega_4^-(q)$ and $O_p(B)$ splits into the direct sum of two X-modules C and $D \cong O_p(B)/C$.

By Sylow's theorem we may assume that the element x is contained in K. Then the normalizer N of $\langle x \rangle$ in X' is contained in $C \times N_K(\langle x \rangle)$. Let $X_1 = O_p(B) \cap X$. It is easy to see that X_1 and $X \cap K$ act semiregularly on Ω. However, we claim that there is a non-trivial element i in $X \setminus X_1$ which fixes a non-singular 1-space. To show this we calculate C. Take a standard basis $\{e_i, f_i \mid 1 \leqslant i \leqslant 4\}$ for V. Then we may assume $W = \langle e_i \mid 1 \leqslant i \leqslant 4 \rangle$. Let $x^2 + ax + b$ be an irreducible $\mathrm{GF}(q)$-polynomial of degree 2 with solution ω (thus in particular $b \neq 0$). Then we consider W as $\mathrm{GF}(q^2)$-vector space by setting $\omega e_i = e_{i+1}$, for $i = 1, 3$. Let x_2, x_3 be the elements of K fixing W as well as the subspace $W' = \langle f_i \mid 1 \leqslant i \leqslant 4 \rangle$ of V and which act on W as the $\mathrm{GF}(q^2)$-matrices

$$\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Let $c = (c_{ij})$ be a non-trivial matrix in C. Then, as $c \in O_p(B)$, it acts trivially on W as well as on V/W and

$$c_{4+ii} = 0 \quad \text{for } 1 \leqslant i \leqslant 4, \quad \text{and} \quad c_{4+ij} = -c_{j+4i} \quad \text{for } 1 \leqslant i < j \leqslant 4.$$

As $[c, x_2] = 1$, we obtain furthermore

$$c_{52} = 0 \quad \text{and} \quad c_{64} = (ac_{63} - c_{53})/b,$$

and $[x_3, c] = 1$ gives

$$c_{74} = 0 \quad \text{and} \quad c_{54} = c_{63}.$$

This determines completely the subgroup C.

Assume that $\overline{X}' = \mathrm{soc}(\overline{X}) \cong \mathrm{L}_2(q^2)$. Observe that we may assume $X^{\infty} \leqslant K$: If $q \neq 3$, then all the complements to D in $H := D : X^{\infty}$ are conjugate by [1, Corollary 4.5]. If $q = 3$, then H

is contained in a subgroup of T isomorphic to $3^4 : (\mathrm{A}_6 \times 4).2$ and also in the latter group we see that X^∞ is conjugate to a subgroup of K.

Thus and as all the elements of order p of the stabilizer of C in $\overline{B}$ are conjugate, in both cases, if $\overline{X}' \cong \mathrm{L}_2(q^2)$ or if $\overline{X}' \cong \mathrm{A}_5$, we may choose $i \in X$ such that $i = cy$ with $c \in X_1$, $c \neq 1$, and such that y is a p-element in $X \cap K$ which acts on W as the $\mathrm{GF}(q^2)$-matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Let $c = (c_{ij})$. Then $C_V(i) = \langle e_1, e_2, -c_{53}e_3 - c_{54}e_4 + f_3, -c_{54}e_3 - c_{64}e_4 + f_4\rangle$.

If $c_{53} = 0$, then the non-singular vector $c_{54}e_3 + (a/b)c_{54}e_4 + f_4$ is fixed and otherwise, the non-singular vector $c_{53}e_3 + c_{54}e_4 + f_3$ is fixed by i.

Thus, if q is even, then X does not act regularly on the orbit Ω of non-singular points of V. Let q be odd. Then there are two orbits Ω_1 and Ω_2 under the action of T on the set of non-singular points of V which consists precisely of the points $\langle v\rangle \leqslant V$ with $Q(v)$ a square or a non-square where Q is the quadratic form on V left invariant by T. Note, for some $0 \leqslant k_1, k_2 < p$ the elements $c^{k_1}y$ and $c^{k_2}y$ fix an element in Ω_1 and Ω_2, respectively. This proves the assertion of the lemma also for q odd. □

Next we determine the structure of a regular subgroup X.

Lemma 10.22. *Let $T \cong \Omega_8^+(q)$, $q \in \{2, 4\}$, and let Ω be the set of non-singular 1-spaces of the natural orthogonal module V for T. If X is a subgroup of $\mathrm{Aut}(T)$ which acts regularly on Ω, then $X \cong \mathrm{Aut}(\mathrm{L}_2(q^2)) \cong \mathrm{P\Gamma L}_2(q^2)$.*

Proof. Let T_A be the stabilizer of a non-singular 1-space of V in T and let $A = N_{\mathrm{Aut}(T)}(T_A)$ and $G = TA$, the biggest subgroup of $\mathrm{Aut}(T)$ which acts on Ω, so $|\mathrm{Aut}(T) : G| = 3$. Let X be a regular subgroup of $\mathrm{Aut}(T)$. Then

$$|X| = |\Omega| = q^3\bigl(q^4 - 1\bigr).$$

Let r be the Zsigmondy prime q_4, so $r = 5$ and 17, if $q = 2$ or 4, respectively.

We first show if X is soluble, then $O_r(X) \neq 1$. Assume that X is soluble. Then $O_p(X) \neq 1$ for some prime p. Assume $p \neq r$. Recall that $|X| = 2^3 \cdot 3 \cdot 5$ and $2^6 \cdot 3 \cdot 5 \cdot 17$ if $q = 2$ and 4, respectively. Let R be a Sylow r-subgroup of X. Then $[O_p(X), R] = 1$. Therefore, $[F(X), R] = 1$ and, as X is soluble, $R \leqslant F(X)$, which yields our claim.

Next assume that X is soluble, let S be a Sylow 2-subgroup of X and let Y be the subgroup $O_r(X)$ extended by S. Notice that $C_Y(R)$ is divisible by 2. Thus there is an involution i in $Z(Y)$. Clearly, $W := C_V(i)$ is 4-dimensional totally isotropic; and Y acts on W. Let B be the stabilizer of W in G. Then we get as in the proof of Lemma 10.21 that Y is contained in $C \times Z$ with C the centralizer of R in $O_2(B)$ and Z a subgroup divisible by $2 \cdot r$. Therefore, we get the same contradiction as in the proof of 10.21. This shows that X is not soluble.

If $q = 2$, then clearly $E(X) \cong \mathrm{L}_2(4)$. If $q = 4$ then it follows $E(X) \cong \mathrm{L}_2(16)$ by using Sylow's theorem or by checking the list of simple groups of order less than 10^{25}, see for instance [11, p. 239].

Finally assume that X is not isomorphic to $\mathrm{Aut}(\mathrm{L}_2(q^2))$. Then $Z(X)$ is non-trivial and contains an involution i. Again, as i does not fix a non-singular point, $C_V(i)$ is a maximal totally singular subspace of V. Then, $E(X)$ fixes this subspace and acts faithfully on it. Now we see again as in the proof of 10.21 that there is an involution in X which fixes a non-singular point. This final contradiction shows $X = \mathrm{Aut}(E(X))$, the assertion. □

Observe the following fact.

Lemma 10.23. *Let $T \cong \Omega_8^+(q)$ with $q \in \{2, 4\}$ and let $X \cong \mathrm{P\Gamma L}_2(q^2)$ be a subgroup of G which is regular on the set of non-singular 1-spaces of the natural orthogonal module V for T. Then there is a maximal totally singular subspace which is fixed by X.*

Proof. Let $K = \mathrm{GF}(q^2)$ be the field with q^2 elements and let W be an irreducible KX-submodule of the KX-module $V^K := V \otimes K$. Then according to Steinberg's tensor product theorem W is the sum of tensor products of the natural KX-module and conjugates of it. If $q = 2$, then it follows that there is a 4-dimensional submodule of V^K which is invariant under the Galois group of K over $\mathrm{GF}(q)$. Thus there is a 4-dimensional X-submodule in V. By assumption this submodule has to be totally singular.

Now let $q = 4$ and let s be an element of order 5 in X. As $C_V(s)$ is a non-degenerate subspace of V, it follows by assumption $C_V(s) = 1$. If $W = V^K$, then V^K is the tensor product of the natural KX-module and all the conjugates of it. This implies the contradiction $C_V(s) \neq 1$. Thus there is again a 4-dimensional submodule of V^K which is invariant under the Galois group of K over $\mathrm{GF}(q)$ and the assertion follows also in that case. □

10.3. Orthogonal groups of odd dimension

The next three lemmas will also be needed for the case that T is an orthogonal group of dimension 8. Moreover, they classify Examples (e) and (m) among the orthogonal groups of dimension 7.

Lemma 10.24. *Let $T \cong \mathrm{Sp}_6(q)$, q even, let $T_A \cong \mathrm{G}_2(q)$ be a subgroup of T and let $T \leqslant G \leqslant \mathrm{Aut}(T)$ be such that $G = TN_G(T_A)$. Then $A = N_G(T_A)$ has a complement X in G if and only if $q \in \{2, 4\}$. Further, if there is a complement X, then $X \cong \mathrm{P\Gamma L}_2(q^2)$.*

Proof. Let $q \in \{2, 4\}$. Then there is a regular subgroup in G, see Examples (e) and (m). Let X be such a regular subgroup. Then there is also a subgroup in $\mathrm{Aut}(\Omega_8^+(q))$ which acts regularly on the set of non-singular points of the natural $\mathrm{GF}(q)$-module for $\Omega_8^+(q)$ and which is isomorphic to X, see Examples (f) and (n). Now Lemma 10.22 implies that $X \cong \mathrm{P\Gamma L}_2(q^2)$.

It remains to show that A does not have a complement X in G if $q > 4$. Let $q \geqslant 8$ and assume that there is complement X to A in G. Note that $N_G(T_A)$ is a maximal subgroup of G. So, in particular, G is an extension of T by a Frobenius automorphism e. Moreover,

$$|X| = |G : A| = q^3(q^4 - 1).$$

Let B be a maximal subgroup of G containing X. By Corollary 6.10 we may assume that T is not contained in B. Hence, $G = AB$ is a maximal factorization and according to [31, Theorem A] $T \cap B$ is isomorphic to one of the following groups:

(a) the stabilizer in G of a 1-space of the natural T-module $(V, (,))$;
(b) $\mathrm{O}_6^+(q)$;

(c) $\mathrm{O}_6^-(q)$;
(d) the stabilizer in G of a non-degenerate 2-space of V.

Recall that $\mathrm{O}_6^+(q) \cong \mathrm{L}_4(q):2$ and $\mathrm{O}_6^-(q) \cong \mathrm{U}_4(q):2$. In cases (b) and (c) there is a subgroup of order $|X|$ in B iff $q = 16$, see Lemma 5.7. In that case $O_2(X) \neq 1$, see Lemmas 5.3–5.7. According to the Borel–Tits theorem X is contained in a maximal parabolic subgroup of T, so of a subgroup as in (a).

(a) To exclude this case we embed G into $H = P\Omega_8^+(q)\langle e\rangle$. Let (W, q) be the natural 8-dimensional module for H'. Then we embed G into H such that G does not stabilize a 1-space in W. This implies that G acts irreducibly on W. Let K be the stabilizer in H of a non-singular point U in W, so $K \cong G$. Then $KG = H$, see [31, Table 4], and as $H' \cap K \cap G \cong \mathrm{G}_2(q)$, see [31, (5.15)], we may assume that $K \cap G = A$. It follows that $KX = K(AX) = KG = H$ and $K \cap X \leqslant K \cap G \cap X = A \cap X = 1$, so X is a complement to K in H and acts regularly on the set Λ of non-singular points of W. By assumption X is contained in the stabilizer B in G of a 1-space of the module V. Hence, $O_2(B) \neq 1$ and therefore B fixes a non-trivial subspace of W. As B is transitive on Λ, [31, Theorem A] implies that B stabilizes a maximal totally singular subspace of W. But this contradicts Lemma 10.21.

(d) Thus B is the stabilizer of a non-degenerate 2-space U of V in G. Then $T \cap B \cong \mathrm{SL}_2(q) \times \mathrm{Sp}_4(q)$, see [26, (4.1.3)], and $T \cap A \cap B$ is maximal in $T \cap A$, see [31, (5.2.3b)], which implies that $A \cap B \cong \mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$, by [12, (5.5)]. We need more information about $A \cap B$. Let L be the subgroup of $T \cap A \cap B$ which is generated by central involutions of A. Then $L \cong \mathrm{SL}_2(q)$, see [12, (3.2)(iii)], $C_V(L)$ is a 2-dimensional non-degenerate subspace of V and $W := [V, L]$ is the orthogonal complement, see the proof of (5.4) in [12]. This implies, as A has just one orbit on the 2-dimensional non-degenerate subspaces of V, that $U = C_V(L)$. Moreover, we get that $L_1 := C_{T\cap A\cap B}(L) \cong \mathrm{SL}_2(q)$ acts trivially on W. As $C_B(W) \cong \mathrm{SL}_2(q)$ it follows that $C_B(W) = L_1$, and as X intersects A trivially, we obtain that $X \cap C_B(W) = 1$. Thus X is isomorphic to a subgroup of B^W, which is itself isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{Sp}_4(q))$; a contradiction to Lemma 5.7 ($q = 16$ is not possible, as $|\mathrm{Out}(T)| = 4$ in that case).

Hence there is no counterexample to the statement of the lemma. □

Lemma 10.25. *Let $T \cong \Omega_7(q)$, $q = p^e$ odd, let T_A be a maximal subgroup of T isomorphic to $\mathrm{G}_2(q)$ and let $T \leqslant G \leqslant \mathrm{Aut}(T)$ be such that $G = N_G(T_A)T$. Then there is no complement to $A = N_G(T_A)$ in G.*

Proof. Assume that there exists a complement X to A in G. Then

$$|X| = |G : A| = \frac{1}{2}q^3\left(q^4 - 1\right).$$

As T_A is maximal in T, we may assume that there is a maximal factorization $G = AB$ with $X \leqslant B$, see Corollary 6.10. According to [31, Theorem A] $T \cap B$ is isomorphic to one of the following groups:

(a) the stabilizer of a 1-dimensional singular subspace of the natural T-module $(V, (\,,\,))$;
(b), (c) the stabilizer of a $\pm$-point in V, $T \cap B \cong \mathrm{O}_6^+(q) \cong \mathrm{L}_4(q):2$ or $\mathrm{O}_6^-(q):2 \cong 2^{\cdot}\mathrm{U}_4(q):2$;
(d), (e) the stabilizer of a line of $\pm$ type in V, $T \cap B \cong (\Omega_2^{\pm}(q) \times \Omega_5(q)).2^2$;
(f) $\mathrm{Sp}_6(2)$ or S_9 and $q = 3$.

Exactly as cases (b) and (c) in the proof of 10.24 cases (b)–(e) are not possible (recall $\Omega_5(q) \cong \mathrm{PSp}_4(q)$).

Consider (f). Here we have $q = 3$ and

$$|X| = 3^3 \cdot 2^3 \cdot 5.$$

By application of Sylow's theorem and the theorem of Burnside on p-complements we see that $O_3(X) \neq 1$. Hence the Borel–Tits theorem yields that X is contained in a maximal parabolic subgroup P of G and by [31, Theorem A] it follows that P is as in (a).

In case (a) we obtain exactly the same contradiction as we did for $T \cong \mathrm{Sp}_6(q)$, which shows the assertion. □

Notice that the two previous lemmas prove Theorem 4.

Lemma 10.26. *Let $T \cong \mathrm{P}\Omega_7(q)$, q odd, and let $T \leqslant G \leqslant \mathrm{Aut}(T)$. Let Ω_i be the set of totally singular spaces of dimension i of the natural $\mathrm{GF}(q)T$-module V, for $i \in \{1, 2, 3\}$. Then there is no subgroup of G which acts regularly on Ω_i, for $i \in \{1, 2, 3\}$.*

Proof. Assume that there is a subgroup X of G which acts regularly on Ω_i, for some $i \in \{1, 2, 3\}$. Let $A = G_x$ be the stabilizer of an element $x \in \Omega_i$ and let B be a maximal subgroup of G containing X. Again by Corollary 6.10 we may assume that T is not a subgroup of B. Then by [31, Theorem A] one of the following holds:

(a) $q = i = 3$;
(b) $i = 1$, $B \cap T \cong \mathrm{G}_2(q)$;
(c) $i = 3$, $q > 3$, $B \cap T \cong \Omega_6^-(q) : 2$.

In case (a), $|\Omega| = 2 \cdot 5 \cdot 7$, so X is soluble and has an element of order 35 in contradiction to [11, pp. 110, 111].

Case (b). Let $i = 1$, $B \cap T \cong \mathrm{G}_2(q)$. Then $A \cap B$ is a parabolic subgroup of B. By Lemma 6.3 it follows that $B = (A \cap B)X$, but there is not such a factorization, see [31, Theorem B].

Case (c). Let $i = 3, q > 3$ and $B \cap T \cong \Omega_6^-(q) : 2 \cong 2.\mathrm{U}_4(q) : 2$, see [10] or [26, (4.1.6)]. Let $\{d, e_j, f_j \mid 1 \leqslant j \leqslant 3\}$ be a standard basis of V. Then B is the stabilizer of a $-$ point in G, say $B = G_{\langle v \rangle}$ with $v = e_1 + f_1$. Suppose that $x = \langle e_1, e_2, e_3 \rangle$. Then $A \cap B$ is a subgroup of the stabilizer B_y of the 2-dimensional totally isotropic subspace $y = x \cap v^\perp = \langle e_2, e_3 \rangle$ of V in B. As

$$|X| = |G : A| = \left(q^3 + 1\right)\left(q^2 + 1\right)(q + 1),$$

$A \cap B$ is in fact a subgroup of index $q + 1$ in B_y. This shows that $B_y \cap T$ is the stabilizer of a line in $B \cap T$ of the natural module for $(B \cap T)' \cong 2.\mathrm{U}_4(q)$ and that $A \cap B$ contains $(B_y \cap T)^\infty \cong q^{1+4}.\mathrm{SL}_2(q)$. Therefore, B_y is the only maximal subgroup of $B \cong \mathrm{U}_4(q) : 2$ which contains $A \cap B$. Now Lemma 6.3 implies that B has a factorization $B = B_y C$ for some subgroup C of B, so $q = 3$, see [31, Theorem A], in contradiction to $q > 3$. This proves the lemma. □

Next we prove Theorem 5.

Proof of Theorem 5. Assume that there is a subgroup X of G which acts regularly on Ω_i for some $i \in \{1, \ldots, n\}$. By Lemma 10.26 we have $n \geqslant 4$.

Let $A = G_x$ be the stabilizer of an element $x \in \Omega_i$ and let B be a maximal subgroup of G containing X. Again, we may assume B does not contain T, see Corollary 6.10. Let $\{d, e_j, f_j \mid 1 \leqslant j \leqslant n\}$ be a standard basis of V. Then B is the stabilizer of a $-$ point in G, say $B = G_{\langle v \rangle}$ with $v = e_1 + \alpha f_1$, for an appropriate $\alpha \neq 0$, and $i = n$, see [31, Tables 1, 2, 3]. We are going to generalize the contradiction which we found in case (c) of the previous lemma. We have $B \cap T \cong \Omega_{2n}^-(q).2$, see [26, (4.1.6)] and $A \cap B$ is a subgroup of index $q+1$ in the stabilizer B_Y in B of a maximal totally singular subspace of the natural B-module W, see [31, (3.4.1)]. Notice that the subgroup B_Y does not stabilize any non-maximal totally singular subspace of W. Thus, as $A \cap B$ contains a Sylow p-subgroup, where $q = p^e$, of B', every maximal subgroup D of B which contains $A \cap B$ is a parabolic subgroup of B and therefore the stabilizer of a maximal totally singular subspace of W. According to [31, Theorem A] there is no factorization $B = XD$ in contradiction with Lemma 6.3. □

10.4. The orthogonal groups of dimension 8

Now we embark upon proving Theorem 6. First we consider again the special case that Ω is the set of non-singular points of the natural module for T.

Proposition 10.27. *Let $T \cong \Omega_8^+(q)$ and let V be the natural orthogonal module for T. Further, let Ω be an orbit in the action of T on the set of non-singular* 1*-spaces of V and let $T \leqslant G \leqslant$* Aut(T) *such that G acts on Ω. If there exists a subgroup X in G which is regular on Ω, then X fixes a maximal totally singular subspace.*

Proof. If $q = 2$ or 4, then the assertion follows with Lemmas 10.21–10.23. Thus assume $q \neq 2, 4$ and that there is a subgroup X of G which acts regularly on Ω. Let A be the stabilizer of a non-singular 1-space of V in G. Clearly, we may assume that $G = TN_{\mathrm{Aut}(T)}(T \cap A)$. Then $T \cap A \cong \mathrm{P}\Omega_7(q)$, see [26, (4.1.6)], $G = AX$ and

$$|X| = tq^3\left(q^4 - 1\right)$$

with $t = 1/(2, q-1)$. Let B be a maximal subgroup of G containing X. By Corollary 6.10 we may assume that T is not contained in B. Hence $G = AB$ is a maximal factorization, and according to [31, Theorem A] B is the stabilizer of a maximal totally singular subspace of V or one of the following holds:

(a) B is absolutely irreducible on V and $T \cap B \cong \mathrm{P}\Omega_7(q)$;
(b) B is the normalizer of a field extension of $\mathrm{GF}(q)$ and $T \cap B \cong (q+1)/a.\mathrm{U}_4(q).2^b$ with $a = (q+1, 4)$ and $b = 1$ if q is even and $2^b = a$ if q is odd (see [26, (4.3.18)]);
(c) B is the stabilizer of a subspace decomposition of V, $q \neq 2$, and $T \cap B \cong \mathrm{GL}_4(q).2$ if q is even and $T \cap B \cong ((q-1)/2).\mathrm{L}_4(q).2^a$ with $2^a = 2((q-1)/2, 2)$ if q is odd, see [26, (4.2.7)];
(d) B is the stabilizer of a tensor decomposition of V and $T \cap B \cong (\mathrm{PSp}_2(q) \times \mathrm{PSp}_4(q)).2$ and q is odd;
(e) B is absolutely irreducible on V and $T \cap B \cong \mathrm{P}\Omega_8^-(q^{1/2})$;
(f) B is absolutely irreducible on V, $T \cap B \cong \Omega_8^+(2)$ and $q = 3$.

In (a) $T \cap B \cong \mathrm{P}\Omega_7(q)$ and X is a complement to $A \cap B \cong \mathrm{G}_2(q)$ in B which contradicts Lemmas 10.24 and 10.25.

Also in case (c) we get a factorization which does not exist: assume that B leaves invariant a subspace decomposition of V. Then by Lemma 5.7 $q = 16$ and $T \cap B \cong \mathrm{GL}_4(16):2$. Again we may assume $G = N_{\mathrm{Aut}(T)}(B)T$ and therefore, $G = T:\langle f\rangle$, with f a field automorphism of T of order 4. Moreover,

$$T \cap A \cap B \cong \mathrm{GL}_3(16):2,$$

see [31, (3.6.1b)]. Let $\overline{B} = B/F(B)$. Then $\overline{(A \cap B)}\overline{X} = \overline{B}$, which yields $X \leqslant D \leqslant B$ with $\overline{D} \cap \overline{B}' \cong \mathrm{Sp}_4(16)$, see [31, Theorem A]. Then $A \cap D$ fixes a non-singular subspace of the natural module for D, see [31, (3.1.2)]. We obtain a contradiction as there is no such factorization $D = (A \cap D)X$ for D, see [31, Theorem A].

Cases (b) and (e) are not possible by arithmetic reasons:

(b) Assume B is the normalizer of a field extension. Then $G = N_{\mathrm{Aut}(T)}(B)T = T:\langle f\rangle$, f a field automorphism of T. From Lemma 5.7 we derive that $q = 16$, so

$$T \cap B \cong \big(\mathbb{Z}_{17} \times \mathrm{U}_4(16)\big):2 \quad \text{and} \quad |X| = 2^{12}\big(2^{16}-1\big).$$

Consider $\overline{B} = B/F(B)$. Then $\overline{B}$ is an almost simple group with $\overline{B}'' \cong \mathrm{U}_4(16)$. Clearly, the factorization of B, $B = (A \cap B)X$, implies a factorization $\overline{B} = \overline{(A \cap B)}\overline{X}$ of $\overline{B}$. Let D be a maximal subgroup of B containing X. Then, also $\overline{B} = \overline{(A \cap B)D}$. Of course, $A \cap B$ is a subgroup of the stabilizer in B of a non-singular 1-dimensional subspace $\langle v\rangle$ of the module V considered as a module for B, see also [31, (5.15)]. This implies that $\overline{D}$ is either the stabilizer P of a totally singular line in the B-module V or $\overline{D} \cap \overline{B}'' \cong \mathrm{Sp}_4(16)$, see [31, Theorem A].

Assume the first. Then $X^\infty \cong \mathrm{L}_2(2^8)$ and $|X : X^\infty| = 16$. As the normalizer of X^∞ in $\overline{P \cap T}$ is isomorphic to $X^\infty:2$, see for instance [26, (4.18)], and as $|F(B)| = 17$, it follows that $|P : P \cap T| = 8$ in contradiction to $G < \mathrm{Aut}(T)$.

Hence $\overline{D} \cap \overline{B}'' \cong \mathrm{Sp}_4(16)$. This embedding can be seen as follows. Let $\{a_1, a_2, b_1, b_2\}$ be a standard basis of the B-module V and let U be the GF(16)-span of this basis. Then U equipped with the unitary form of V restricted to U becomes a symplectic space and D its automorphism group. By [31, (3.3.7a)] $\overline{A} \cap \overline{D} \leqslant \overline{A \cap B} \cap \overline{D}$ is contained in the stabilizer of a 2-dimensional non-degenerate subspace of U, and

$$\overline{T \cap A \cap D} \cong \big(\mathrm{Sp}_2(16) \times \mathbb{Z}_{17}\big).2^a \quad \text{with } a \in \{0, 1\}.$$

Clearly, $D = (A \cap D)X$ and $(A \cap D) \cap X = 1$, which is not possible by the theorem of Lagrange.

(e) Here $T \cap B \cong \mathrm{P}\Omega_8^-(q^{1/2})$, with q a square, and $T \cap A \cap B \cong \mathrm{G}_2(q^{1/2})$, see [31, (5.15)], and as q_4 divides the order of $T \cap X$, it follows that $T \cap X$ is a subgroup of the maximal subgroup D of B with $T \cap D \cong \mathrm{P}\Omega_4^-(q).2 \cong \mathrm{L}_2(q^2).2$ (see [31, Table 2.5] and [26, (4.3)]). Notice that $T \cap D$ acts irreducibly on V, which implies that D is a subgroup of $\mathrm{Aut}(T \cap D)$. As $q \neq 4$ it follows that $|D|$ is not divisible by q^3, a contradiction.

If $q = 3$, then $|X| = 2^3 \cdot 3^3 \cdot 5$. The theorem of Sylow implies $O_3(X) \neq 1$. Hence X lies in a maximal parabolic subgroup of G by the theorem of Borel and Tits. So the assertion holds in case (f).

It remains to consider (d). The embedding of $T \cap B$ in T can be described as follows (see [31, 2.2.5]). Let V_1 be a 2-dimensional symplectic space over GF(q) with standard basis $\{c, d\}$ with respect to a symplectic form $(\,,)_1$, and let V_2 be a 4-dimensional symplectic space over $\mathrm{GF}(q)$ with standard basis $\{a_1, a_2, b_1, b_2\}$ with respect to a symplectic form $(\,,)_2$. We identify V with $V_1 \otimes V_2$, setting

$$Q(u_1 \otimes u_2) = 0 \quad \text{and} \quad (u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)_1 (u_2, v_2)_2$$

for $u_i, v_i \in V_i$ $(i = 1, 2)$. We may also take for $i = 1, 2$,

$$e_i = c \otimes a_i, \quad f_i = d \otimes b_i \quad \text{and} \quad e_{2+i} = c \otimes b_i, \quad f_{2+i} = d \otimes -a_i.$$

Let B_1 and B_2 be the factors $\mathrm{PSp}_2(q)$ and $\mathrm{PSp}_4(q)$ of $T \cap B$, respectively. As X is regular, it follows that $\overline{X}$ is transitive on the set of non-degenerate 2-dimensional subspaces of V_2 where $\overline{B} = B/B_1$. [31, Theorem A] implies that $\overline{X}$ has a subgroup isomorphic to $\mathrm{L}_2(q^2)$ with $q \in \{2, 4\}$. As $X \cap B_2$ is a normal subgroup of X and as the Zsigmondy prime q_4 divides the order of X, also $X \cap B_2$ contains a subgroup isomorphic to $\mathrm{L}_2(q^2)$. This implies that X induces on V_1 a 2-group. Therefore X fixes a 1-dimensional subspace of V_1 and a maximal totally singular subspace of V. □

Proposition 10.27 and Lemmas 10.21–10.23 imply the following corollary.

Corollary 10.28. *Let $T \cong \Omega_8^+(q)$ and let V be the natural orthogonal module for T. Further, let Ω be an orbit in the action of T on the set of non-singular 1-spaces of V and let $T \leqslant G \leqslant \mathrm{Aut}(T)$ such that G acts on Ω. If there exists a subgroup X in G which is regular on Ω, then*

(a) *X fixes a maximal totally singular subspace*;
(b) $X \cong \mathrm{P\Gamma L}_2(q^2)$; *and*
(c) $q = 2, 4$.

In the remainder of this section we prove Theorem 6.

Proof of Theorem 6. Assume that there is an example not listed above. First we discuss some special cases when $q = 2, 3$ or 4 and then we discuss the general situation. Let $A = G_x$ be the stabilizer of an element x of Ω in G. Then A is a maximal subgroup of G and X is a complement to A in G. Let B be a maximal subgroup of G which contains X. In all the cases we are considering, except for (b.b) and (b.c), T acts primitively on Ω. Hence, by Corollary 6.10 we may always, except for (b.b) and (b.c), assume that B does not contain T. Thus, in all cases apart from (b.b) and (b.c), $G = AB$ is a maximal factorization of G.

Let (V, Q) be the natural module for T with non-degenerate quadratic form Q of $+$ type. We need to recall the geometry for T. There are two orbits under the action of T on the set of maximal totally singular subspaces of V, call them $\mathcal{O}_1$ and $\mathcal{O}_2$ [37, (11.61)]. Two maximal subspaces are in the same orbit if and only if their intersection is of even dimension [37, (11.61)]. The elements of the building $\mathcal{B}$ for T are the totally singular subspaces of dimension i, for $1 \leqslant i \leqslant 2$, and the elements in the two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$. Two elements are incident in $\mathcal{B}$ if one is contained in the other or if both are of dimension 4 and they intersect in a subspace of dimension 3, see [37, p. 173].

(a) $q = 2$. Here we consider the following special factorizations:

(a.a) $A \cap T \cong (3 \times \Omega_6^-(2)).2$ and $B \cap T \cong \mathrm{A}_9$;
(a.b) $A \cap T \cong \Omega_4^+(4) : 2^2$ and $B \cap T \cong \mathrm{Sp}_6(2)$.

(a.a) If $A \cap T \cong (3 \times \Omega_6^-(2)).2$, then $|X| = |G : A| = 2^5 \cdot 5 \cdot 7$ in contradiction to the fact that there is no subgroup of this size in B, see [11, p. 37].

(a.b) Assume $A \cap T \cong \Omega_4^+(4) : 2^2$ and $B \cap T \cong \mathrm{Sp}_6(2)$. Then $|X| = |G : A| = 2^6 \cdot 3^3 \cdot 7$. We may assume that $B \cap T$ is the stabilizer of a non-singular 1-space $\langle v \rangle$ of V. Then

$$T \cap A \cap B \cong 2 \times \Omega_3(4) \cong 2 \times \mathrm{A}_5$$

and X is a complement to $A \cap B$ in B. We read from the list of subgroups of $\mathrm{Sp}_6(2)$ that $T \cap X \cong \mathrm{G}_2(2)$, see [11, p. 46].

We are going to show that some 3-elements of X are in the same conjugacy class as the 3-elements in $A \cap B$ which is a contradiction to Lemma 6.2. According to the Atlas the elements of order 3 of X are of type 3B and 3C in B (in the Atlas-notation) [11, pp. 46, 47].

Let r be an element of order 3 in $A \cap B$. We claim that r is of type 3C. Let $\{a_1, a_2, b_1, b_2\}$ be a standard basis of V considered as a natural GF(4)-module for $A \cap T$ and let $(\,,\,)_{\mathrm{GF}(4)}$ be the form on V left invariant by $A \cap T$. Let ω be a generator of the multiplicative group of GF(4). Without loss of generality we may assume that $v = \omega(a_1 + b_1)$. Then r fixes $\langle a_1 + b_1 \rangle_{\mathrm{GF}(4)}$ elementwise and stabilizes the subspace $\langle a_1 + b_1, a_2, b_2 \rangle_{\mathrm{GF}(4)}$ of V which is perpendicular to $a_1 + b_1$ with respect to the form $(\,,\,)_{\mathrm{GF}(4)}$. As $A \cap B$ induces on $\langle a_2, b_2 \rangle_{\mathrm{GF}(4)}$ a group isomorphic to $\mathrm{SL}_2(4) : 2$, we may assume that r acts on $W = \langle a_1 + b_1, a_2, b_2 \rangle_{\mathrm{GF}(2)}$ (that is the matrix of r with respect to the basis $\{a_1 + b_1, a_2, b_2\}$ has its entries in GF(2)). Hence r fixes the maximal totally singular subspace W of the 6-dimensional symplectic space $U = v^\perp / \langle v \rangle_{\mathrm{GF}(2)}$. Then according to the Atlas r is of type 3C in $\mathrm{Aut}(W) = B \cap T$, see [11, p. 46], which is a contradiction to Lemma 6.2.

(b) $q = 3$. Here we discuss the following special cases.

(b.a) $A \cap T \cong \Omega_8^+(2)$;
(b.b) A is the stabilizer of a flag in the building $\mathcal{B}$ consisting either of a point and a maximal totally singular subspace, or of two maximal totally singular subspaces, from $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively, and $T : 2 \leqslant G$;
(b.c) $A \cap T \cong 2^6 : \mathrm{A}_8$ and $T : 2 \leqslant G$.

(b.a) Suppose $A \cap T \cong \Omega_8^+(2)$. Then G/T is isomorphic to a subgroup of S_3, so we may assume that G contains a triality automorphism, and

$$|X| = |G : A| = 3^7 \cdot 13,$$

which implies that $|O_3(X)| \geqslant 3^6$. So by the Borel–Tits theorem X is contained in a maximal parabolic subgroup $B = P$ of G. As 13 divides $|X|$ it is a subgroup of type $A_3(3)$. Moreover, there is $Y \times Z \leqslant A$ with $|Y| = 3$ and $Z \cong \mathrm{P}\Omega_5(3)$, see [11, p. 85]. Again by Borel–Tits $Y \times Z$ lies in a parabolic Q with $Y \leqslant O_3(Q)$. As Q is of type $A_3(3)$, as well, and as A is triality-invariant, we may assume $Q = P$.

Note that $B \cong 3^6 : \mathrm{L}_4(3)$, see the Atlas [11, p. 140], and that $O_3(B)$ is the 6-dimensional orthogonal module, call it W, for

$$\overline{B} = B / O_3(B) \cong \mathrm{L}_4(3) \cong \mathrm{P}\Omega_6^+(3).$$

It follows that $A \cap B = Y : K_Y$ with $Y = \langle v \rangle$ a non-singular 1-space of W and K_Y its stabilizer in a complement K of $O_3(B)$ in B. As $W \cap X$ does not contain a non-singular vector and since x acts on $W \cap X$, where x is an element of order 13 of X, it follows that $W_1 := W \cap X$ is a 3-dimensional totally singular subspace of W.

We are going to derive a contradiction by finding an element in $A \cap X$. Let $\{e_i, f_i \mid 1 \leqslant i \leqslant 3\}$ be a standard basis of W such that

$$W_1 = \langle e_1, e_2, e_3 \rangle \quad \text{and} \quad W_2 = \langle f_1, f_2, f_3 \rangle.$$

We may assume $v = e_1 + f_1$ and that x lies in K. Then $M := N_K(\{W_1, W_2\})$ is isomorphic to $\mathrm{L}_3(3) : 2$ and, as $|K|_{13} = 13$, all subgroups of order 13 of K are conjugate in K and we may assume $x \in M$.

Then $N_M(\langle x \rangle) \cong \mathrm{Frob}(13:6)$ and $M' \cap N_M(\langle x \rangle) \cong 13:3$. Hence, while x acts on W_1 as the linear map x, it acts on W_2 as x^{-1}. As $M' \cap N_M(\langle x \rangle) \cong \mathrm{Frob}(13:3)$, the elements x and x^{-1} are not conjugate in M' and therefore, W_1 and W_2 are not isomorphic modules for $\langle x \rangle$. Hence, W is the direct sum of two non-isomorphic $\langle x \rangle$-modules.

Next we determine the intersection $X \cap K$. As x acts fixed point freely on $O_3(B)$, the subgroup K contains beside x also $X_2 := N_X(\langle x \rangle) \cong \mathrm{Frob}(13:3)$.

We claim that $X_2 = X \cap K$. Assume $X_2 < X \cap K$. Then X_2 is a complement to W_1 in X; so $|X_2|_3 = 3^4$. Therefore, $|K \cap A|_3 = 3^4$ and $|K|_3 = 3^6$ imply that X_2 and A intersect non-trivially in contradiction to our assumption that X and A intersect trivially. This shows that $X_2 = X \cap K$.

Clearly, X is contained in the extension E of W by the stabilizer in K of the maximal singular subspace W_1 of W. It is

$$E \cong 3^6 : 3^3 : \mathrm{L}_3(3).$$

Let $R = O_3(E \cap K)$, so R is elementary abelian of order 27, $W : R = O_3(E)$ and $O_3(X)$, which is a group of order 3^6, is contained in $W : R$, and it is the direct sum of two non-isomorphic $\langle x \rangle$-modules.

Let t be an element of order 3 in X_2. Then $|C_D(t)| = 3$ for all X_2-modules

$$D \in \{W_1, W_2, O_3(X)/W_1, R\}.$$

Let t be chosen such that $C_{W_2}(t) = \langle f_1 \rangle$ and let $a \in O_3(X) \setminus W_1$ such that $[t, a] = 1$. Then $a = f_1 u$ with $u \in C_R(\langle t \rangle)$. In E we see that $[C_{W_2}(t), C_R(t)] = 1$. As R acts trivially on W_1, it follows that $(e_1 + f_1)u$ is an element in A, which is also contained in X, and we obtain a contradiction to our assumption that X is a complement to A in G. Thus A is not as in (b.a).

(b.b) Here $A \cap T \cong 3^{3+6} : \mathrm{L}_3(3)$, $B \cap T \cong \Omega_8^+(2)$ and

$$|X| = |G : A| = 2^8 \cdot 5^2 \cdot 7,$$

see [31, Theorem A] and [11, p. 140]. It follows that X is soluble, but this is not possible as G has no subgroup of order $5^2 \cdot 7$.

(b.c) $A \cap T \cong 2^6 : \mathrm{A}_8$ and $T : 2 \leqslant G$. Then

$$|X| = |G : A| = 3^{10} \cdot 5 \cdot 13.$$

By Burnsides Normal p-complement theorem X has a normal 5-complement and $|O_3(X)| \geqslant 3^9$. By the Borel–Tits theorem X is contained in a maximal parabolic P of G, which is clearly of type $A_3(3)$. But this is not possible as P has no subgroup isomorphic to X.

(c) $q = 4$. Now we rule out the following possible special factorization.

(c.a) $A \cap T \cong (\mathrm{L}_2(16) \times \mathrm{L}_2(16)).2^2$, $T:2 \leqslant G$ and $B \cap T \cong \Omega_7(4)$. Here,

$$|X| = |G:A| = 2^{14} \cdot \bigl(2^{12}-1\bigr)\bigl(2^4-1\bigr).$$

According to [31, 2.5], we see that there is no subgroup of size $|X|$ in B.

(d) For the rest of the proof q is an arbitrary prime power, and $d := (2, q-1)$. It remains to work on the following cases, see [31, Theorem A].

(d.a) Ω is a T-orbit of totally singular subspaces of dimension i, for $i = 1$ or 4;
(d.b) Ω is the set of non-degenerate 2-dimensional subspaces in V of $+$ type;
(d.c) Ω is the set of non-degenerate 2-dimensional subspaces in V of $-$ type;
(d.d) $A \cap T \cong (\mathrm{PSp}_2(q) \times \mathrm{PSp}_4(q)).2$ and $q > 2$;
(d.e) $A \cap T \cong \mathrm{P}\Omega_8^-(q^{1/2})$ with q a square.

(d.a) Let Ω be a T-orbit of totally singular i-spaces, for $i = 4$; application of the triality automorphism to G will then cover the general case. Here

$$|X| = |G:A| = \bigl(q^3+1\bigr)\bigl(q^2+1\bigr)(q+1)$$

and $B \cap T$ is either isomorphic to $\mathrm{P}\Omega_7(q)$ or to $\hat{}((q+1)/d \times \mathrm{P}\Omega_6^-(q)).2^d$. If $B \cap T \cong \mathrm{P}\Omega_7(q)$, then $A \cap B \cap T$ is the stabilizer in B of a maximal totally singular subspace of the natural module for B. By Lemma 10.26 $A \cap B \cap T$ does not have a complement X in B. Hence, $B \cap T \cong \hat{}((q+1)/d \times \mathrm{P}\Omega_6^-(q)).2^d$ is the stabilizer of a 2-dimensional subspace of $-$ type of V in G. Again, $A \cap B \cap T$ is contained in a parabolic subgroup of B. Let E be a maximal subgroup of B which contains X. Then $B = (A \cap B)E$ and according to [31, Theorem A] one of the following holds:

(d.a.a) E is the stabilizer in B of a non-isotropic point in the natural module for $B \cong \hat{}\mathrm{GU}_4(q):2$;
(d.a.b) $q = 2$ and $E \cong 3^3.\mathrm{S}_4$;
(d.a.c) $q = 3$ and $E \cong \mathrm{L}_3(4)$.

(d.a) is not possible as if (d.a.a) and (d.a.b) holds, then the group E is not divisible by the order of X and if (d.a.c), then E does not contain a subgroup of size $|X|$.

(d.b) Ω is the set of non-singular 2-spaces of $+$ type. Then

$$|X| = |\Omega| = q^6\bigl(q^3+1\bigr)\bigl(q^2+1\bigr)(q+1).$$

According to [31, Table 4] $q > 2$, $G = T:2$ if $q = 3$, and $T \cap B \cong \mathrm{P}\Omega_7(q)$. Let $\{e_i, f_i \mid 1 \leqslant i \leqslant 4\}$ be the standard basis of V. Applying the triality automorphism to G we may assume that A is the stabilizer of the set $\{U_1, U_2\}$ of two maximal totally isotropic subspaces with $U_1 = \langle e_i \mid$

$1 \leqslant i \leqslant 4\rangle$ and $U_2 = \langle f_i \mid 1 \leqslant i \leqslant 4\rangle$ and that B is the stabilizer of the non-singular point $\langle v\rangle$ with $v = e_1 + f_1$. Then

$$(A \cap B)^\infty / Z\big((A \cap B)^\infty\big) \cong \mathrm{L}_3(q).$$

Clearly, $B = (A \cap B)X$ and if $A \cap B \leqslant D \leqslant B$ or $X \leqslant E \leqslant B$, then we get an exact factorization of D or E, respectively. Moreover, notice, that the stabilizer of a $-$ point of the natural module W for B does not contain a factor isomorphic to $\mathrm{L}_3(q)$ and that $A \cap B$ does not fix a totally singular subspace of W, see [31, (3.6.1b)]. As, moreover, the Zsigmondy prime q_6 divides $|X|$, we derive from [31, Theorem A] and Theorem 3 that $q = 3$ and $E \cong \mathrm{S}_9$ or $\mathrm{Sp}_6(2)$. In both cases the order of E is not divisible by $|X|$. Hence case (d.b) does not hold.

(d.c) Assume Ω is the set of non-singular 2-spaces of $-$ type. Here

$$|X| = |\Omega| = \frac{1}{2}q^6\big(q^3 - 1\big)\big(q^2 + 1\big)(q - 1).$$

By [31, Table 4] we have either $T \cap B \cong \mathrm{P}\Omega_7(q)$, B is a maximal parabolic of type $A_3(q)$ or $q = 2$ and $B \cap T \cong \mathrm{A}_9$.

Assume first that B is a parabolic subgroup. Let $\overline{B} = B/O_p(B)$. Lemma 6.4 implies that we get a non-trivial factorization of $\overline{B}^\infty / Z(\overline{B}^\infty) \cong \mathrm{L}_4(q)$ such that one of the factors is divisible by the product of the two Zsigmondy primes q_4 and q_3, which is impossible (see [31, Theorem A]). If $B \cap T \cong \mathrm{A}_9$, then we get a contradiction as there is no exact factorization $B = (A \cap B)X$ with $|X| = 1120$ (there does not exist a subgroup of size $|X|$ in S_9, see also [38]).

Thus $B \cong \mathrm{P}\Omega_7(q)$. Applying the triality automorphism to G, we may assume that A is the normalizer of a field extension and that B is the stabilizer of a non-singular point in V. Then $A \cap B$ is contained in the stabilizer of a non-isotropic 1-space in the natural unitary module for A, which implies, as $|B : A \cap B| = |X|$, that

$$(A \cap B)^\infty / Z\big((A \cap B)^\infty\big) \cong \mathrm{U}_3(q).$$

Now we argue as in the case (d.b). Again we get an exact factorization $B = (A \cap B)X$ and [31, Theorem A] implies here, as $A \cap B$ is not contained in a maximal parabolic subgroup of B of type $A_3(q)$ and as the Zsigmondy primes q_3, q_4 divide $|X|$, that $q = 3$ and $X \leqslant E \leqslant B$ with $T \cap E \cong \mathrm{S}_9$ or $\mathrm{Sp}_6(2)$ or $T \cap E \cong \mathrm{L}_4(3) : 2$. In the first two cases $|X|$ does not divide $|E|$ and in the last case there is no subgroup of size $|X|$ in E, see [11, p. 69].

(d.d) and (d.e). We have

$$|X| = |G : A| = \frac{1}{2}q^7\big(q^6 - 1\big)\big(q^2 + 1\big) \quad \text{or}$$
$$|X| = |G : A| = q^6\big(q^4 - 1\big)\big(q^3 + 1\big)(q + 1), \quad \text{respectively,}$$

and in both cases by [31, Table 4] $B \cap T \cong \mathrm{P}\Omega_7(q)$. As B does not possess a subgroup of size $|X|$, see for instance [18] or [24], we obtain a contradiction and there is no counterexample to the assertion of the theorem. $\square$

10.5. The unitary groups

Finally we prove Theorems 7 and 8.

Proof of Theorem 7. Assume the assumptions of the theorem. Notice that T as well as $\mathrm{Aut}(T)$ act primitively on Ω. This implies that there is a subgroup $T \leqslant G^\star \leqslant G$ which admits a maximal factorization $G^\star = AB$ such that $X \leqslant B$ and $A = G^\star_x$ for some $x \in \Omega$, see Corollary 6.10. By [31, Theorem A] an almost simple group with socle isomorphic to $\mathrm{U}_n(q)$ admits a maximal factorization either if n is even and $i = n/2$ or if $n = 3$ and $q \in \{3, 5, 8\}$ or if $(n, q) = (4, 2)$ and $i = 2$ or $i = 1$ and $(n, q) = (4, 3)$ or $(9, 2)$.

Assume $n = 3$. If $q = 3$ or 5, then $|\Omega| = 2^2 \cdot 7$ or $2 \cdot 3^2 \cdot 7$ and B is isomorphic to a subgroup of $\mathrm{L}_2(7) : 2$ or S_7, respectively, see [31, Table 3]. In both cases B does not contain a subgroup of order $|\Omega|$. Hence $q = 8$ and (G, X) are as in Example (o), see Lemma 10.17.

Now assume $n = 4$ and $q = 3$. Then B is isomorphic to a subgroup of $\mathrm{L}_3(4) : 2^2$, see [31, Table 3], and $|\Omega| = 2^3 \cdot 5 \cdot 7$ and $2^4 \cdot 7$, if $i = 1$ or 2, respectively. Again B does not contain a subgroup of order $|\Omega|$, see for instance [11, p. 23].

Next assume $(n, q) = (9, 2)$ and $i = 1$. Then $T \cap B$ is isomorphic to J_3. Hence we obtain a non-trivial factorization of J_3 or $\mathrm{Aut}(\mathrm{J}_3)$, see Lemma 6.3, which does not exist by [31, Theorem C].

Thus if we assume that the assertion is false, then we obtain with Lemma 10.2 that n is even and $i = n/2$. According to [31, Table 1], B is the stabilizer of a non-isotropic 1-space $\langle v \rangle$ of V, so $B \cap T \cong \hat{}\mathrm{GU}_{n-1}(q)$, i.e. $(B^\infty/Z(B^\infty)) \cong \mathrm{U}_{n-1}(q)$ with $|Z(B^\infty)| \leqslant q + 1$, and B acts faithfully on $W = v^\perp$. Moreover, $A \cap B$ is a subgroup of index $q + 1$ in the stabilizer B_Y in B of some maximal totally isotropic subspace Y of W. As $B = (A \cap B)X$, $(n - 1, q)$ and i are as listed above or B is soluble or the factorization is not proper. Assume the latter. Then either $Z(B)(A \cap B) = B$ or $Z(B)X = B$. As B acts transitively on Ω, the first equation does not hold, and it is easy to see that $|A \cap B| > |Z(B^\infty)|$, so $Z(B)X = B$ does not hold as well. If B is soluble, then $n = 4$, $q = 2$ and the pair (T, X) is as in Example (a).

Finally assume the first case, that is $(n - 1, q)$ and i are as listed above. As n is even, we get $n = 4$ and, as we already showed $(n, q) \neq (4, 3)$, that $q \in \{5, 8\}$. Assume $q = 5$. Then $T \cap B \cong 3^{\cdot}\mathrm{U}_3(5) : 3$ and, as $|X| = (q + 1)(q^3 + 1)$, that $|X| = 2^2 \cdot 3^3 \cdot 7$ and that X is a subgroup of $D \leqslant B$ with $DZ(B)/Z(B) \cap E(B/Z(B)) \cong \mathrm{A}_7$, which is impossible. Hence $q = 8$ and (G, X) are as in Example (q), see Lemma 10.17. □

Proof of Theorem 8. Suppose the assumptions of the theorem and assume that there is an example different from Examples (a), (o), (p), (q) or (r).

By Theorem 7 we may assume that Ω is not a set of totally isotropic subspaces of V and by Lemma 10.2 that $(n, q) \neq (4, 2)$. Moreover, as the possible factorizations of $\mathrm{U}_3(8)$ yield the Examples (o) and (p), see the list of examples and [31, Theorem A], we may also assume that $(n, q) \neq (3, 8)$. Let A be the stabilizer in G of an element in Ω. According to [31, Theorem A] either $T \in \Lambda$, with

$$\Lambda = \{\mathrm{U}_3(3), \mathrm{U}_3(5), \mathrm{U}_4(3), \mathrm{U}_6(2), \mathrm{U}_9(2), \mathrm{U}_{12}(2)\}$$

and $A \cap T$ is listed in [31, Table 3] or $n = 2m$ for some natural number $m \geqslant 2$ and $A \cap T$ is a group in Δ where Δ is the set of groups isomorphic to one of the following

$$\mathrm{PSp}_{2m}(q); \qquad \hat{}\mathrm{SL}_m(4) : 2, \quad m \geqslant 3 \text{ and } q = 2; \qquad \hat{}\mathrm{SL}_m(16).3.2 \quad \text{and } q = 4.$$

In all cases T acts primitively on Ω. Let B be a maximal subgroup of G which contains X. Then by Corollary 6.10 we may assume that T is not in B.

Assume that $n = 2m$, $m \geqslant 2$, and that $A \cap T$ is in Δ. Then B is the stabilizer of a non-singular 1-space of the natural $\mathrm{GF}(q)T$-module V, see [31, Table 1], and therefore $B \cong \hat{}\mathrm{GU}_{n-1}(q)$, see [26, (4.1.4)]. Clearly, Lemma 6.3 implies that we obtain a non-trivial factorization of $B'/Z(B') \cong \mathrm{U}_{n-1}(q)$. Then application of [31, Theorem A] to this situation yields one of the following: $(n-1, q) = (3,3), (3,5), (3,8)$ or $(9,2)$. First assume $q \neq 2$. Then $n = 4$, $A \cap T \cong \mathrm{PSp}_4(q)$ and

$$|\Omega| = |G : A| = \frac{(2, q-1)}{(4, q+1)} q^2\big(q^3+1\big).$$

So, if $q = 3, 5$ or 8, then

$$|\Omega| = 2 \cdot 3^2 \cdot 7, 2 \cdot 3^2 \cdot 5^2 \cdot 7, 2^6 \cdot 3^3 \cdot 19,$$

respectively. By [31, Theorem A], for $q \in \{3,5,8\}$, there is no factorization $\mathrm{U}_3(q) = CD$ such that one of the factors is divisible by $|\Omega|$. Thus our assumption was false and $q = 2$. So $(n-1, q) = (9, 2)$. Then either $A \cap T \cong \mathrm{PSp}_{10}(2)$ or $\hat{}\mathrm{SL}_5(4) : 2$, and $|\Omega| = |G : A|$ is divisible by $2^{20} \cdot 19$ in both cases. By [31, Theorem A] there is no factorization of $\mathrm{U}_9(2)$ such that one of the factors is divisible by $2^{20} \cdot 19$. Therefore, $A \cap T$ is not in Δ. Thus T is isomorphic to a group in Λ.

Assume $T \cong \mathrm{U}_3(3)$ or $\mathrm{U}_3(5)$. Then $|G : T| \leqslant 2$, $A \cap T \cong \mathrm{L}_2(7)$ or A_7 (see [31, Table 3]) and

$$|X| = |\Omega| = 2^2 \cdot 3^2 \quad \text{or} \quad 2 \cdot 5^2,$$

respectively. In the first case $A \cap T$ as well as $X \cap T$ contain an involution of the only class of involutions of T, see [11, p. 14], and in the second case A intersects all the classes of involutions of G, see [11, pp. 34, 35]. Thus in both cases we obtain a contradiction to Lemma 6.2.

Assume $T \cong \mathrm{U}_4(3)$, but case (r) does not hold. Then by [31, Table 3] $T \cap A \cong \mathrm{PSp}_4(3)$, $T \cap B \cong \mathrm{L}_3(4)$, and G is a subgroup of $T : 2^2$, see [11, p. 52]. Moreover,

$$|X| = |\Omega| = 2 \cdot 3^2 \cdot 7.$$

This yields a contradiction as there is no proper subgroup in $\mathrm{L}_3(4)$ which is divisible by $3^2 \cdot 7$, see [11, p. 23].

Finally assume $T \cong \mathrm{U}_6(2), \mathrm{U}_{12}(2)$ or $\mathrm{U}_9(2)$. Then $T \cap B \cong \mathrm{U}_5(2), \mathrm{U}_{11}(2)$ or $O_2(B) \neq 1$ and $(T \cap B)/O_2(B) \cong \mathrm{U}_7(2)$, respectively, see [31, Table 3] and [26, (4.18), (4.1.4)]. Hence, we obtain either a non-trivial factorization of $\mathrm{U}_5(2), \mathrm{U}_{11}(2)$ or of $\mathrm{U}_7(2)$ or $O_2(B)(A \cap B) = B$ or $O_2(B)X = B$ by Lemma 6.3. Theorem A of [31] implies that one of the two latter cases holds and that $T \cong \mathrm{U}_9(2)$. We get a contradiction in both cases: as $(A \cap B) \cong 2^{2+4}.(3 \times \mathrm{S}_3)$ by [31, (5.2.12)], and as $|X|_3 = |G : A|_3 = 3^4 < |\mathrm{U}_7(2)|_3 = 3^6$. $\square$

11. An application

In [13] Etinghof, Gelaki, Guralnick and Saxl constructed a biperfect semisimple Hopf algebra. Given a finite group G and two subgroups G_1, G_2 of G such that G_2 is a complement to G_1 in G, then following Takeuchi [36], one can construct a semisimple Hopf algebra $H(G, G_1, G_2)$ of dimension $|G|$ from these data. If G_1 and G_2 are selfnormalizing perfect subgroups of G, then $H(G, G_1, G_2)$ is biperfect [13]. The authors gave an example of such an algebra, namely $H_1 = H(\mathrm{M}_{24}, \mathrm{L}_2(23), 2^4 : \mathrm{A}_7)$. In Question 3.4(2) they ask whether there exist biperfect Hopf algebras of dimension less than $|\mathrm{M}_{24}|$. It seems that their example is unique. The example H_1 is unique for G an alternating, a sporadic or an exceptional group of Lie type, see Theorems 1–3, and it seems that there is no example with G a classical group.

12. Some remarks on Burnside-groups

Let G be a group which acts faithfully and primitively on a set Ω. In 1911 W. Burnside proved that if (G, Ω) contains a cyclic regular subgroup of prime power order, then G is doubly transitive or of prime degree [8, p. 343]. Later I. Schur could extend this result to all cyclic groups: if G contains a cyclic regular subgroup, then G is doubly transitive or of prime degree [35]. In order to prove this fact Schur introduced the powerful theory of S-rings. In 1935 H. Wielandt [39] generalized this theorem to abelian groups with a cyclic Sylow subgroup: if G contains an abelian regular subgroup which has a cyclic Sylow subgroup, then G is doubly transitive or of prime degree. Later in his book [40] Wielandt introduced the notion of a Burnside-group. A group X is called a *Burnside-group* (or short a *B-group*) if each primitive permutation group which contains a regular subgroup isomorphic to X is necessarily 2-transitive. Hence the above theorem states in fact that every abelian group which has a cyclic Sylow subgroup is a B-group, see also [40, Chapter IV, § 25].

The classification of primitive permutation groups which contain an abelian regular subgroup by Li, see [27, Theorem 1.1], implies the following:

Corollary 12.1. *Let X be an abelian group such that $|X|$ is not a prime power. Then X is a B-group if and only if there are no subgroups $X_1, \ldots, X_l$, $l \geqslant 2$, with $|X_1| = \cdots = |X_l|$ such that $X = X_1 \times \cdots \times X_l$.*

Corollary 12.2. *Let X be an abelian p-group. Then one of the following holds.*

(a) *X is a B-group*;
(b) *$X = X_1 \times \cdots \times X_n$ with $|X_i| = |X_j| > 2$ and $n \geqslant 2$*;
(c) *X is a regular subgroup of a primitive but not doubly transitive group of affine type.*

Clearly, (b) and (c) are not exclusive. Note that already Wielandt showed that a group listed in (b) is never a B-group [40, Theorem 25.7]. In this context see also [20].

In [40, p. 49] he poses the following problem:

Which elementary abelian 2-group X of order 2^p, p a prime, is a B-group?

If $2^p - 1$ is not a prime in addition, then X is not a B-group: Let x be an element of $\mathrm{Aut}(X)$ whose order is a Zsigmondy prime, then x acts irreducibly on X and X extended by $\langle x \rangle$ is not a doubly transitive permutation group. Wielandt asks whether the fact that $2^p - 1$ is a prime, is a sufficient condition for X to be a B-group. According to Corollary 12.2 if we want to answer this question, then we need to determine the irreducible subgroups H of $G = \mathrm{GL}_p(2)$ which are not transitive on $V^{\#}$, V the natural $\mathrm{GL}_p(2)$-module. Checking the list of subgroups of $\mathrm{L}_n(q)$ for $n \leqslant 11$ by Kleidman [24], we see that there are no subgroups fulfilling these requirements for $n \leqslant 11$. S. Linton [28] did run through the whole GAP library of Brauer characters and did not find an irreducible subgroup which is not transitive, as well. Assume there exists an irreducible subgroup which is not transitive. Consider the classification of the maximal subgroups of the classical groups by Aschbacher, see [2]. As p is a prime, it follows that H is not contained in a geometric subgroup of G.

Also the complete classification of B-groups which are p-groups for p odd is still open. But there are more results. For instance R.D. Bercov, see [7], showed the following.

Theorem 12.3. [7] *Let $X = X_1 \times X_2$ such that X_1 and X_2 are cyclic of order p^a and p^b with $a > b$ and p an odd prime. Then X is a B-group.*

For a discussion on Burnside-groups see also [40, Chapter IV, § 25].

It seems to be an interesting question to study the irreducible subgroups of the automorphism group $\mathrm{GL}(V)$ of an elementary abelian group V of order p^n with n and $(p^n - 1)/(p - 1)$ prime numbers which have more than one orbit on the set of non-trivial vectors $V^\#$ and thereby pushing forward the classification of the p-groups which are B-groups.

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